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APPLICATION OF GAUSS-KRONROD QUADRATURE INTEGRATION SCHEME WITH ENRICHEMENT IN SIMULATIONS OF HISTORY-DEPENDENT MATERIALS

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Abstract. *In finite element modeling of problems involving history-dependent materials, storing of the computational history of strains and stresses at the integration points is essential for accurate results. After each converged load-step, plastic strains are stored and accumulated at the integration points. This process is usually accomplished by employing Gauss quadrature integration scheme throughout the elements. However, a problem arises when the element is enriched with a new node which is common in contact problems. Enrichment of an element introduces a higher order term in the element shape functions associated with the nodes located on the interface. Thus, the order of interpolation has to be increased in the direction where the node is inserted. In hyper-elasticity, this process is usually accomplished by increasing the order of the Gauss integration scheme, thus introducing new integration points. However, this is not the case in simulating a history-dependent material, where the accumulated plastic strains data at the integration points must be preserved before enrichment. In this paper, Gauss-Kronrod quadrature rule is applied to extend the applicability of the Enriched Discontinuous Galerkin Approach (EDGA) for plasticity. Gauss-Kronrod quadrature inherits Gauss points locations and provides an additional set of integration points that lays between the original Gaussian quadrature. When the enrichment is introduced and Gauss-Kronrod integration points are inserted, the values of the plastic variables at the new integration points can be computed by interpolation/extrapolation from the existing values using the element shape functions. To verify the implantation of the Gauss-Kronrod integration scheme, the patch test is used for a single Q4 element under uniaxial tension. The results showed that the Q4 element with Gauss-Kronrod quadrature integration points passes the patch test and reflects a constant stress distribution exactly, and also a consistent deformed shape with exact solution.*

1 INTRODUCTION

In structural engineering and many other fields, computational models of contact, with history-dependent material behavior, are in high demand. The use of the finite element method (FEM) without accurate consideration of the involved contact interactions and accurate material behavior may cause inaccurate results leading to failure in these systems. For two bodies in contact discretized using different finite element meshes or in case of large sliding, the nodes from one body do not coincide with those from the other, resulting in a non-conforming mesh (NCM). The use of NCMs, however, presents several numerical issues; the main challenge with such discretization is to ensure compatibility of the kinematic and traction fields along the non-conforming interface.

The EDGA addresses this challenge by implementing a local enrichment and interface stabilization procedure based on the Discontinuous Galerkin formulation, to enable a two-pass approach in enforcing contact conditions that preserves the weak continuity of surface tractions without introducing dual interface fields [1-4]. In this approach the local enrichment can be enforced at all nodes along the interface. The enrichment in the element introduces a higher order in the element shape function associated with the nodes located on that interface. Thus, the order of interpolation has to be increased in the direction where the node is inserted. Solving this problem in hyperelasticity is usually accomplished by increasing the order of the Gauss integration scheme, thereby introducing new integration points.

Equations (1-3) define the shape functions of the inserted node 5 as illustrated in Fig. 1. To preserve the

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interpolatory nature of the finite element basis and its partition of unity property, the shape functions associated with existing nodes are modified as follows, where $\widetilde{N}^a(\xi,\xi^P)$ is the shape function of the enriched node, N^a_{Q4} is the shape function of Q4 element for a=1,…,4 and $\widetilde{N}^a$ are the modified (enriched) element shape functions.

$$\widetilde{N}^P(\xi,\xi^P) = \frac{1}{2}(\xi_2+1)\frac{(\xi_1+1)(\xi_1-1)}{(\xi^P+1)(\xi^P-1)} \tag{1}$$

$$\widetilde{N}^a = N^a_{Q4} - N^a_{Q4}(\xi^P)N^P \tag{2}$$

$$\widetilde{N}^a(\xi,\xi^P) = N^a(\xi) - N^a(\xi^P)N^a(\xi,\xi^P) \tag{3}$$

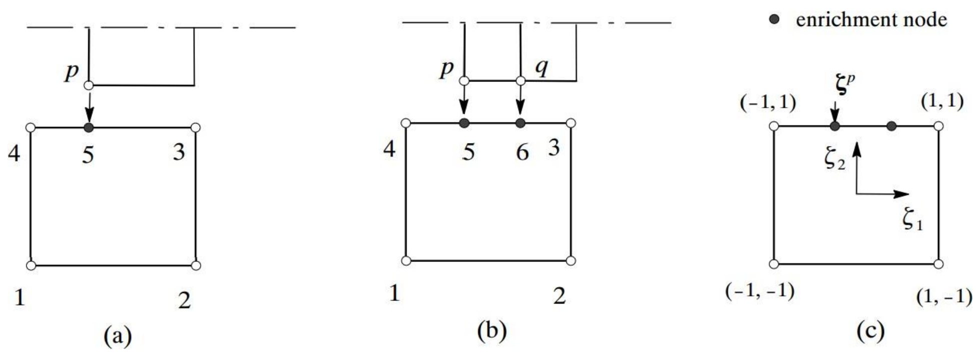

Figure 1. Local enrichment of the interface element: (a) single node, (b) multiple nodes, and (c) added node reference in the parent domain [1].

In this study, the EDGA developed for the coupling of the NCM is extended to model large-deformation contact problems between bodies with inelastic constitutive behavior [5]. The enrichment process could be problematic, in which the plastic strains are stored and accumulated at the Gauss points after each converged load step, the computational history at the integration points before enrichment must be preserved. Therefore, a progressive integration rule such as the Gauss-Kronrod quadrature can be used alternatively. Important studies regarding contact problems involving plasticity and its application in soil-structure interaction problems can be found in [6-10].

2 GAUSS-KRONROD QUADRATURE

The Gauss-Kronrod quadrature inherits Gauss point locations and provides an additional set of integration points interlaced between the original Gaussian quadrature. To compute the number and locations of additional Kronrod points required to evaluate the integration accurately, we start by computing an estimate of the integral with the original Gauss quadrature. Then, we re-compute it using two sets of points combined; the original Gauss points set and Gauss-Kronrod set. The difference between the values of the two sets gives an estimate of the error in the results. The derivation of the Gauss- Kronrod formula is similar to standard Gauss quadrature. The Gauss-Kronrod quadrature and its implementation are explained through the following example.

Consider the enriched element shown in Fig. 2; assuming an enrichment of the top surface $\zeta^2 = 1$, which introduces a quadratic term in ζ^1 in the element shape functions associated with the nodes located on this interface, while the order of interpolation with respect to ζ^2 remains the same. Therefore, the integration rule order has to be increased in the direction of ζ^1. For the use of this element in contact simulations, two different sets of Gauss- Kronrod integration points are needed. The first set of points is used inside the element as illustrated in Figure 2, where the stresses and plastic strains are computed to find the internal forces and the stiffness of the element. In addition, Gauss-Kronrod integration points are needed on the interface as illustrated in Figure 3, to be used for the stabilization terms, where the stresses and plastic strain are computed.

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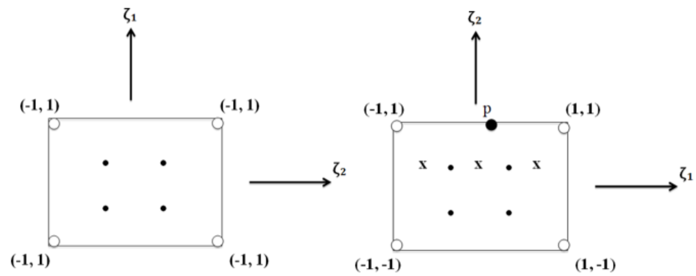

Figure 2. Q4 element with Gauss quadrature integration points inside (left) and the enriched element with Gauss-Kronrod integration points (right).

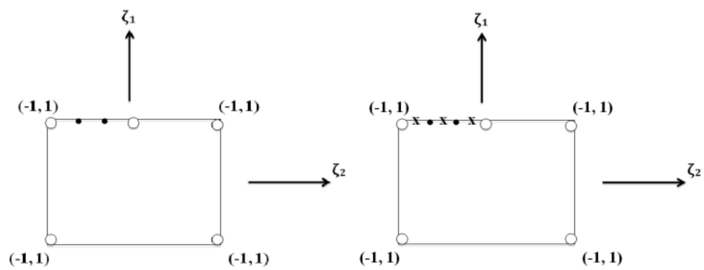

Figure 3. Q4 element with Gauss quadrature integration points at the interface (left) and the enriched element with Gauss-Kronrod integration points (right).

Computing the Gauss-Kronrod integration points and weights involves evaluating the integral using a *N*-point Gauss quadrature according to Equation (4), where w_k and x_k are the locations and weights of the original Gauss points.

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{N} w_k f(x_k) \qquad (4)$$

Evaluating the integral again using a *2N+1* Gauss-Kronrod quadrature. The location of the *N* points from step one is reused. The locations and corresponding weights of *N+1* additional points are recalculated, as well as the weights associated with the *N* Gauss points:

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{N} z_k f(x_k) + \sum_{j=1}^{N+1} m_j f(y_j) \qquad (5)$$

The set of nodes x_k is precisely the one used in the original Gauss quadrature. All the other *3N+2* parameters z_k, m_j, and y_j are chosen such that Equation (5) reaches its maximum degree of accuracy. The value of the plastic variables, at the new integration points, will be computed by interpolating/extrapolating from the existing values using the element shape functions. Table 1 lists the values of the locations and weights for the Gauss-Kronrod integration points *i* including the Gauss quadrature for exact integration of a cubic function N=2.

Table 1: Gauss-Kronrod Quadrature Locations and Weights for N=2

i	**Kronrod** ξ_1	**Kronrod** ξ_2	**Kronrod** w_i	**Gauss** w_i
1	-0.92582009977	0.57735026918	0.19797979798	---
2	-0.57735026918	0.57735026918	0.49090909090	1
3	0	0.57735026918	0.62222222222	---
4	0.57735026918	0.57735026918	0.49090909090	1
5	0.92582009977	0.57735026918	0.19797979798	---

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3 NUMERICAL RESULTS

Gauss-Kronrod quadrature implantation is first verified by using a simple example involving a single Q4 element with enrichment under uniaxial compression. Next, with the presence of enrichment (EDGA), the Gauss-Kronrod quadrature integration scheme is verified for the cases of plasticity and large deformation. Case studies are presented to verify this technique. For the verification of the Gauss-Kronrod Integration scheme, a single Q4 element shown in Figure 5 has an elastic modulus (E) =30,000 ksi, Poisson's ratio (υ) = 0.3, and yield stress (f_y) = 60 ksi. is used under uniaxial compression. The applied distributed load P equals 65 ksi. these conditions are identical to the well-known finite element patch test. and the expected solution is a constant pressure profile in the element. We assume plain-strain conditions. Fig. 5 (left) shows that the element is enriched at the bottom surface, which introduces a higher order term in the element shape functions associated with the enriched node. The Gauss-Kronrod quadrature integration points are used along the enriched side as shown in Fig. 6 (right). The results in Fig. 6 show that the Q4 element with Gauss-Kronrod quadrature integration points passes the patch test and reflects a constant stress distribution exactly. Fig. 7 shows the Q4 element deformed shape, which is also consistent with the exact solution.

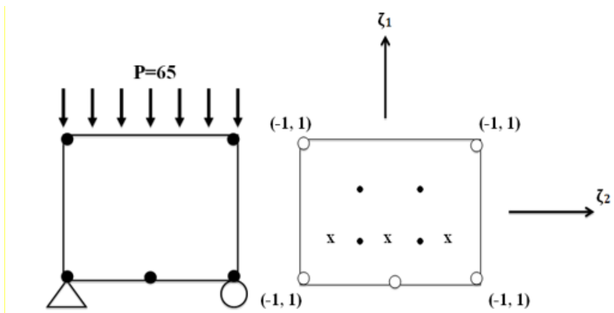

Figure 5. Patch test for Q4 element with enrichment (left) and the Gauss-Kronrod quadrature integration points in the parent element (right)

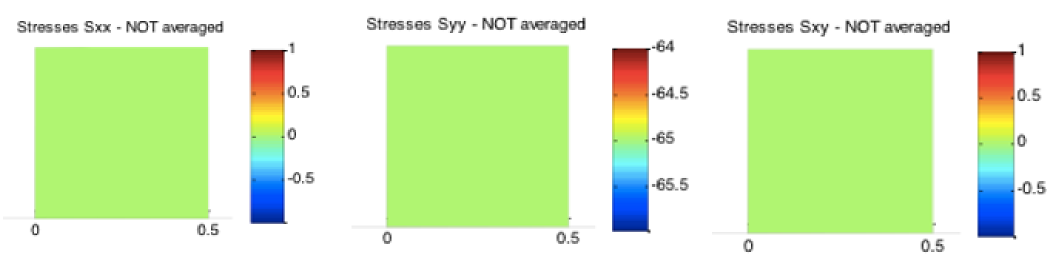

Figure 6. Stress distributions for Q4 element with Gauss-Kronrod integration points.

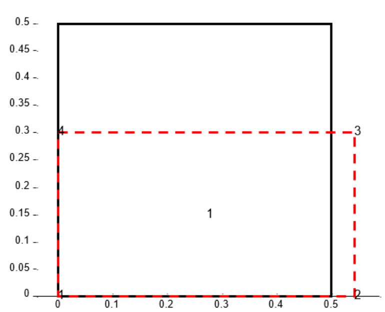

Figure 7. Deformed shape for Q4 element with Gauss-Kronrod integration points.

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Next, Gauss-Kronrod integration scheme is applied to contact problems where EDGA is applied and with the presence of elasto-plastic material with the Von Mises yield criterion, where E = 30,000 ksi, υ = 0.3, and f_y = 60 ksi. Distributed load P = 70 ksi is applied to the top free surfaces of both bodies. The domains are discretized using Q4 elements under plane strain condition. Figs.8 (left) and 9 (left) show the solution obtained without treating the non-conforming interface. The results show inaccurate pressure distribution, and the deformed shape displays similar inaccuracies at nodes 8, 9, and 10. Plasticity in this problem is activated, since the applied load is greater than f_y. The plastic strain at the integration points is around 0.00965. Figs. 8 (right) and 9 (right) show the solution obtained by applying the EDGA for plasticity using Gauss- Kronrod quadrature integration points inside the element to find the tangent and the internal forces and at the interface for the stabilization terms. The results of this case show that the deformed configuration and the stress distributions pass the patch test up to machine precision.

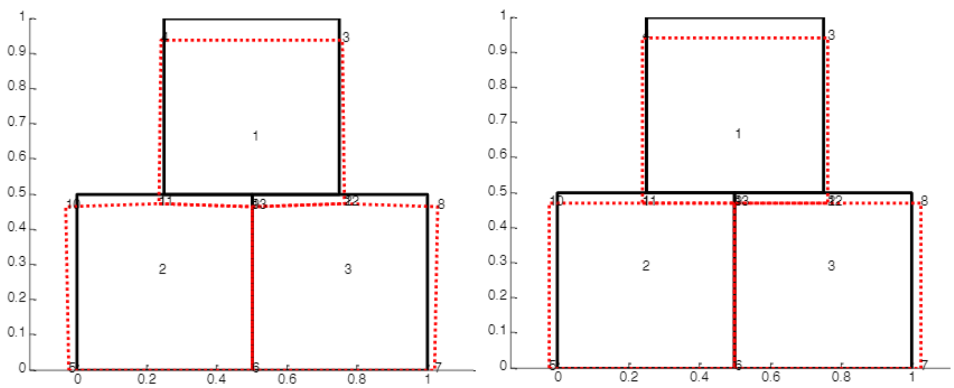

Figure 8. Contact patch test for the elasto-plastic case (Von Mises): deformed shape without EDGA (left) and with EDGA extension for plasticity (right).

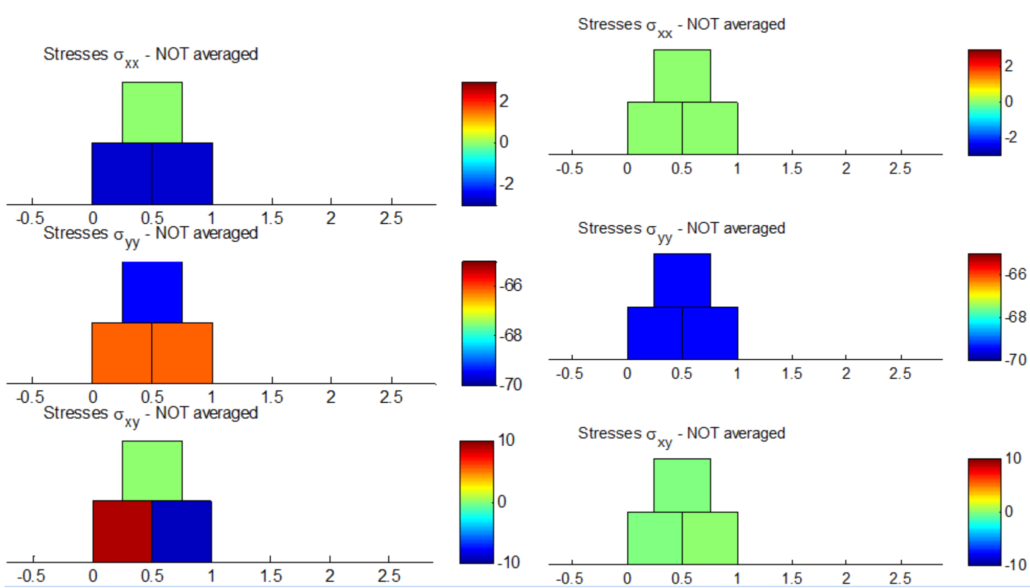

Figure 9. Contact patch test for the elasto-plastic case (Von Mises): stress field without EDGA (left) and with EDGA extension for plasticity (right).

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The patch test is also performed in a different configuration as shown in Fig. 10 to show that the Gauss-Kronrod quadrature is able to handle elements with non-constant Jacobian. An elasto-plastic material with the Von Mises yield criterion is used with E = 30,000 ksi, υ = 0.3, and f_y = 50 ksi. A distributed load of P = 60 ksi is applied to the top free surfaces of both bodies. The domains are discretized using Q4 elements under plane strain conditions. Fig. 11 (left) shows the solution obtained without treating the non-conforming interface. The results show inaccurate pressure distribution and the deformed shape is again inconsistent with the result expected with constant pressure. Plasticity in this problem is again activated, since the applied load is greater than f_y. The plastic strain at the integration points is around 0.00895. Fig. 11 (right) show the solution obtained by applying the EDGA for plasticity using Gauss-Kronrod quadrature integration points inside the element to find the tangent and the internal forces and at the interface for the stabilization terms. As obtained in the previous two cases, the results of this case show that the deformed configuration and the stress distributions pass the patch test up to machine precision.

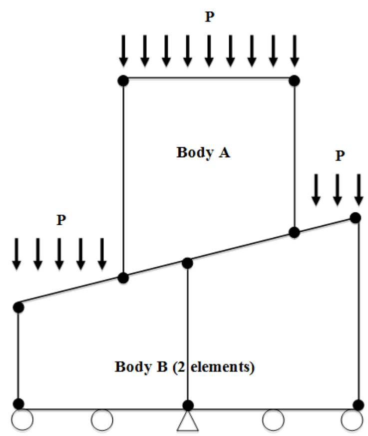

Figure 10. Contact patch test for the elasto-plastic case (Von Mises)

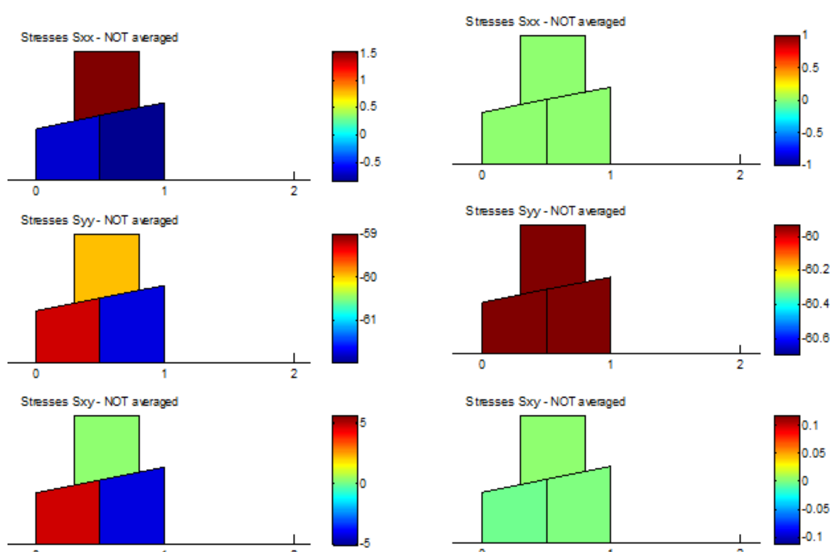

Figure 10. Contact patch test for the elasto-plastic case (Von Mises): stress field without EDGA (left) and with EDGA extension for plasticity (right).

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4 CONCLUSIONS

The following conclusions can be drawn based on the finding of this research:

1. Simulations of contact problems need to take into account both the interface properties and the plastic behavior of the materials. To ensure accurate results in problems involving history-dependent materials, storing the computational history of strains and stresses at integration points is a necessity.
2. In order to increase the order of integration without losing material history at existing integration points, extending the EDGA to problems involving plasticity requires applying the Gauss-Kronrod quadrature rule to inherit Gauss points locations and provides an additional set of integration points that lays between the original Gaussian quadrature. The plastic variables at the new integration points can be computed by interpolation/extrapolation from the existing values using the element shape functions.
3. Using the patch test for a single Q4 element under uniaxial tension, implantation of the Gauss-Kronrod Integration Scheme is verified in terms of. constant stress distribution exactly and also a consistent deformed shape with exact solution.
4. Implantation of Gauss-Kronrod quadrature is verified for contact problems of different configurations using contact patch test where the EDGA is applied at the interface for elasto-plastic material.

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