Electromagnetics II

Transmission Lines

Prepared By:

Nihad Dib, Assoc. Prof
Chap. 11

Transmission Lines

- TL's consist of 2 or more conductors, used to connect source to load.
- Commonly used in power distribution & communications.

Examples:

- Parallel plate TL
- 2-wire line
  - Power lines
  - TV antenna
- Coaxial line
  - Used in Labs, & TV antennas
- Microstrip line
  - Integrated CKTs

- Will use circuit theory to study TL's.
- Will study TEM wave propagation inside TL's.
- TEM means \( E_z = H_z = 0 \), where \( z \) is along the TL.
  \( \vec{E} \) & \( \vec{H} \) are transverse to the direction of propagation.
We describe any T.L. by its line parameters:

1. \( R \) (resistance per unit length) \( \Omega/m \)
2. \( L \) (inductance "") \( H/m \)
3. \( G \) (conductance "") \( S/m \)
4. \( C \) (capacitance "") \( F/m \)

These parameters are distributed along the line (not lumped)

- \( G \) is due to the lossy dielectric between the 2 conductors — current flow between the 2 conductors.
  - If \( \sigma_d = 0 \Rightarrow G = 0 \)

- \( R \) is the ac resistance of the conductors
  - If \( \sigma_c = \infty \) (perfect conductors) \( \Rightarrow R = 0 \)

- \( C \) is the capacitance between the 2 conductors

- \( L \) is the external inductance due to the mag. flux around the wires. Internal inductance is negligible at high frequencies due to skin effect.
Note that \( G \neq \frac{1}{R} \)

For any T.L. (under TEM wave)

\[
LC = \mu e \quad \& \quad \frac{G}{C} = \frac{\sigma}{\varepsilon}
\]

- Coaxial line

\[
R = \frac{1}{2 \pi \delta \sigma_c^2} \left( \frac{1}{a} + \frac{1}{b} \right)
\]

\[
L = \frac{\mu}{2 \pi} \ln \frac{b}{a}
\]

\[
G = \frac{2 \pi \sigma}{\ln b/a}
\]

\[
C = \frac{2 \pi \varepsilon}{\ln (b/a)}
\]

- Parallel plate line (planar line)

\[
R = \frac{2}{W \delta \sigma_c^2}
\]

\[
L = \frac{\mu d}{W}
\]

\[
C = \frac{\varepsilon W}{d}
\]

\[
G = \frac{\sigma W}{d}
\]

\( W \gg d \)
11.3 T.L. Equations

- Instead of using $\mathbf{E}$ & $\mathbf{H}$, we will use circuit theory (i.e., $V$ & $I$)

$$V = -\int \mathbf{E} \cdot d\mathbf{l}, \quad I = \oint \mathbf{H} \cdot d\mathbf{l}$$

- Circuit theory is simpler & more convenient.

- Consider an "incremental" length $\Delta z$ of any T.L.

\[V(z,t) + \Delta z, I(z+\Delta z,t)\]

\[\begin{align*}
R \Delta z & \quad L \Delta z \\
G & \quad C \Delta z \\
V(z,t) & \quad V(z+\Delta z,t)
\end{align*}\]

- L-type equivalent circuit

KVL $\Rightarrow$ $V(z,t) = R \Delta z \cdot I(z,t) + L \Delta z \frac{dI}{dt} + V(z+\Delta z,t)$

- $\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = R \cdot I + L \frac{dI}{dt}$

As $\Delta z \to 0$

$$- \frac{\partial V}{\partial z} = R I + L \frac{dI}{dt}$$

KCL $\Rightarrow$ $I(z,t) = I(z+\Delta z,t) + G \Delta z \cdot V(z+\Delta z,t)$

As $\Delta z \to 0$ $\Rightarrow$ $- \frac{\partial I}{\partial z} = G \cdot V + C \frac{\partial V}{\partial t}$
\[
\frac{\partial V}{\partial t} = RI + L \frac{\partial I}{\partial t} \]
\[
- \frac{\partial I}{\partial t} = GV + C \frac{\partial V}{\partial t} \]

General T.L. eq's

Telegrapher's eq's

For time harmonic dependence

\[
V(z, t) = Re \left\{ V_s(z) e^{j\omega t} \right\}
\]
\[
I(z, t) = Re \left\{ I_s(z) e^{j\omega t} \right\}
\]

\[
\Rightarrow \quad - \frac{dV_s}{dz} = (R + j\omega L) I_s
\]
\[
- \frac{dI_s}{dz} = (G + j\omega C) V_s
\]

or

\[
- \frac{dV_s}{dz} = Z I_s, \quad Z = R + j\omega L \quad \text{(series impedance/m)}
\]
\[
- \frac{dI_s}{dz} = \gamma V_s, \quad \gamma = G + j\omega C \quad \text{(shunt admittance/m)}
\]

need to separate \( I_s \) & \( V_s \):

\[
- \frac{d^2 V_s}{dz^2} = Z \frac{dI_s}{dz} = -Z \gamma V_s
\]

wave eq's

\[
\frac{d^2 V_s}{dz^2} = \gamma^2 V_s
\]

propagation const.

\[
\gamma = \alpha + j\beta = \sqrt{Z\gamma}
\]

attenuation const.

phase const.

or

\[
\frac{d^2 I_s}{dz^2} = \gamma^2 I_s
\]

= \sqrt{(R + j\omega L)(G + j\omega C)}
\[ \lambda = \frac{\alpha}{\beta}, \quad \text{and} \quad u = \frac{\omega}{\beta} = f \lambda \]

- Solution of the wave eq's:
  \[ V_z(t) = V_0^+ e^{\gamma z} + V_0^- e^{-\gamma z} \]
  \[ I_z(t) = I_0^+ e^{\gamma z} + I_0^- e^{-\gamma z} \]

- \( V_0^+, V_0^-, I_0^+, I_0^- \) are wave amplitudes (complex in general)

- \( V(z, t) = V_0^+ e^{\gamma z} \cos(\omega t - \beta z) + V_0^- e^{\gamma z} \cos(\omega t + \beta z) \)

\[ \text{wave traveling in } +z \text{ direction} \quad \text{wave traveling in } -z \text{ direction} \]

- Define: Characteristic impedance
  \[ Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R+j\omega L}{\gamma} = \frac{\gamma}{G+j\omega C} \]
  \[ Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{Z}{\gamma}} \]

\[ \text{proof:} \]
  \[ -\frac{dV}{dz} = (R+j\omega L) I \]
  \[ \Rightarrow \quad \gamma V_0^+ e^{\gamma z} - \gamma V_0^- e^{-\gamma z} = (R+j\omega L) I \]
  \[ \frac{\gamma V_0^+}{R+j\omega L} e^{\gamma z} - \frac{\gamma V_0^-}{R+j\omega L} e^{-\gamma z} = I = I_0^+ e^{\gamma z} + I_0^- e^{-\gamma z} \]
  \[ \therefore \quad \frac{\gamma V_0^+}{R+j\omega L} = I_0^+ \quad \text{and} \quad -\frac{\gamma V_0^-}{R+j\omega L} = I_0^- \]
\[ Z_0 = R_0 + jX_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \]  \hspace{2cm} (1)

\[ Y_0 = \frac{1}{Z_0} : \text{characteristic admittance} \]

- **Note:** \( R_0 \) is different from \( R \) (resistance/\( \text{m} \)).

- **Note:** \( Z_0 \) is independent of \( z \). It depends on the parameters \( R, L, G, \) and \( C \). Thus, it depends on the shape of the conductors, dimensions, dielectric properties, and conductors' properties.

For an infinite-length line \( \Rightarrow \) no reflection

\[ \Rightarrow e^{\gamma z} \text{ vanishes} \]

\[ \therefore V(z) = V_0^+ e^{-\gamma z}, \quad I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} \]

In this case,

\[ Z_0 = \frac{V_0^+}{I_0^+} = \frac{V(z)}{I(z)} \]

- In general,

\[ V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \]

\[ I(z) = \frac{V_0^+ - \gamma z}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \]
Two types of losses:

1. Dielectric losses, \( \sigma_d \neq 0 \Rightarrow G \)
2. Conductor losses, \( \sigma_c \neq \infty \Rightarrow R \)

1. Loss less line \(( R = G = 0 )\)

\[
\gamma = \alpha + j\beta = j\omega \sqrt{LC}
\]

\[
\Rightarrow \alpha = 0, \quad \beta = \omega \sqrt{LC}
\]

\[
u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad \text{-- constant}
\]

\[
Z_0 = \sqrt{\frac{L}{C}} = R_0, \quad X_0 = Z_0 \omega
\]

\[
V(z) = V_0^+ e^{-j\beta^2} + V_0^- e^{j\beta^2}
\]

\[
I(z) = \frac{V_0^+}{Z_0} e^{-j\beta^2} - \frac{V_0^-}{Z_0} e^{j\beta^2}
\]

2. Distortion less line \( \left( \frac{R}{L} = \frac{G}{C} \right) \)

\[
\gamma = \alpha + j\beta = \left[ (R + j\omega L)(\frac{RC}{L} + j\omega C) \right]^{1/2}
\]

\[
= \sqrt{\frac{C}{L}} (R + j\omega L)
\]

\[
\Rightarrow \alpha = R \sqrt{\frac{C}{L}} = R \sqrt{\frac{G}{R}} = \sqrt{RG}
\]

\[
\beta = \omega \sqrt{LC} \quad \text{-- same as lossless}
\]

But, \( \alpha \neq 0 \)
\[ u = \frac{w}{\beta} = \frac{1}{\sqrt{LC}} \quad \text{constant, independent of frequency} \]

- \( x \) & \( u \) not \( \text{fr}'s \) of frequency \( \rightarrow \) no distortion

\[ Z_o = \sqrt{\frac{R+j\omega L}{\frac{RC}{L}+j\omega C}} = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}} = R_0 \]

\[ X_o = 0 \]

- \( x = R \sqrt{\frac{C}{L}} = \frac{R}{R_0} \)

- A lossless line is distortionless. But, a distortionless line is not necessarily lossless.

---

- What is \( Z_o \) for a coaxial line?

\[ Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\varepsilon \mu}{2\pi} \frac{\ln (b/a)}{\ln (b/a)}} \]

\[ = \sqrt{\frac{\mu/\varepsilon}{2\pi}} \ln (b/a) \]

\[ = 60 \sqrt{\frac{\mu_0}{\varepsilon_0}} \ln (b/a) \]

- What is \( Z_o \) for a lossless parallel plate line?

\[ Z_o = \sqrt{\frac{\mu d/W}{\varepsilon W/d}} = \eta \frac{d}{W} \]
Oliver Heaviside discovered this nondistorting case in 1887 while studying the performance of long-distance telegraph circuits. At that time, it was known that the maximum rate at which telegraph signals could be transmitted on a line varied inversely with the length of the line, but the reason for this phenomenon was a subject of fierce debate. Heaviside was the first to discover why distortion occurs on lossy lines and theorized that it could be reduced or eliminated on practical transmission lines by adjusting the line parameters so that Equation (11.40) is satisfied. The initial engineering application of Heaviside’s idea to telegraph lines was done by George Campbell and, to a lesser extent, Mischa Auerbach.\(^1\)


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![Diagram of loading coils on a transmission line](image)

Figure 11-10 Loading coils placed periodically on a transmission line to reduce dispersion.

On practical transmission lines, the values of \( R, C, L, \) and \( G \) are usually such that \( RC > GL \). To obtain a distortionless line, either \( RC \) must be decreased, or \( GL \) must be increased. This could be accomplished by increasing the conductor spacing, which increases \( L \) and decreases \( C \). However, that would result in unacceptable conductor spacings. Another unacceptable solution is to add shunt conductance to the line to increase \( G \), which also increases the attenuation constant \( \alpha \). Heaviside proposed raising \( L \) by placing series inductors (called loading coils) periodically along the line, as depicted in Figure 11-10. This simple procedure was an immediate success and is still used on analog telephone links to allow distortionless transmission throughout the voice band (0–3 [kHz]).

Example 11-3

A twisted-pair telephone cable transmission line has the following parameters:

\[
R = 0.107 \, [\Omega/m] \\
L = 543 \, [nH/m] \\
C = 51.3 \, [pF/m] \\
G = 51.0 \, [\mu F/m].
\]

Find the loading-coil inductance that must be added at each kilometer of the line in order to obtain distortionless propagation.

**Solution:**

Using Equation (11.40) and the specified values of \( R, G, \) and \( C \), we find that the required value of the inductance is

\[
L' = \frac{RC}{G} = \frac{0.107 \times 51.3 \times 10^{-12}}{51.0 \times 10^{-12}} = 0.108 \, [H/m].
\]

Using \( L' = L + L_c \), where \( L_c \) is the loading-coil inductance per meter, we find that

\[
L_c = 0.108 \, [H/m] - 543 \, [nH/m] = 108 \, [H/km].
\]

11.3.3 LAUNCHING WAVES ON TRANSMISSION LINES

A wave can be launched on a transmission line simply by attaching a voltage across its terminals. Figure 11-11a depicts such a situation. Here, an independent voltage generator \( V_o(t) \) and a resistor \( R \), are connected to the end of an infinite, lossless transmis-
An air line, \( Z_0 = 70 \Omega \), \( \beta = 3 \text{ rad/m} \), \( f = 100 \text{ MHz} \). Find \( L \) & \( C \).

Since filled with air \( \rightarrow \sigma_d = 0 \rightarrow G = 0 \). 
\( Z_0 \) is real \( \rightarrow \) lossless.

\[ R = G = 0 \]

\[ \beta = \omega \sqrt{LC} \quad , \quad Z_0 = R_0 = \sqrt{\frac{L}{C}} \]

\[ Z_0 \beta = \omega L \quad \Rightarrow \quad L = \frac{Z_0 \beta}{\omega} = 334.2 \text{ nH/m} \]

\[ \frac{Z_0}{\beta} = \frac{1}{\omega C} \quad \Rightarrow \quad C = \frac{\beta}{\omega Z_0} = 68.2 \text{ pF/m} \]

Check: \( LC = \frac{\mu_0}{\varepsilon_0} \)

Here: \( LC = \left( 334.2 \times 10^9 \right) \left( 68.2 \times 10^{12} \right) = 0.228 \times 10^{-16} \)

But, \( \frac{1}{\mu_0 \varepsilon_0} = 0.131 \times 10^{-16} \) ! ! ! !

\( \beta = 3 \) is wrong! It should be

\[ \beta = \omega \sqrt{LC} = \omega \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{2 \times 10^8}{3 \times 10^8} = 2.094 \]

\[ \# \]

Suggested problems:
11. 4 - 6, 10, 15, 18, 21, 24, 26, 28, 30, 31,
39-41, 43
Ex. a 50 Ω - disturbance line, \( \alpha = 0.01 \) dB/m

\( C = 0.1 \) nF/m. Find \( R, L, G \) !!, \( u_p \), ...

\[
\alpha = 0.01 \text{ dB/m} = \frac{0.01}{8.69} = 1.15 \times 10^{-3} \text{ Np/m}
\]

(1) \[ \frac{R}{L} = \frac{C}{C} \quad ; \quad (2) \quad Z_0 = \sqrt{\frac{L}{C}} \quad ; \quad (3) \quad \alpha = R \sqrt{\frac{C}{L}} \]

\[
\therefore (2) \Rightarrow L = Z_0^2 C = 0.25 \text{ mH/m}
\]

(3) \[ R = \alpha \sqrt{\frac{L}{C}} = \alpha Z_0 = 0.057 \text{ Ω/m} \]

(1) \[ G = C \frac{R}{L} = 22.8 \text{ MS/m} \]

(4) \[ u_p = \frac{1}{\sqrt{LC}} = 2 \times 10^8 \text{ m/s} \]

(4) Find percentage to which \( V \) decreases in 1 km!

\[
V(z) = V_0 e^{-\alpha z} = V(z=0) e^{-\alpha z}
\]

\[
\frac{V(z)}{V(z=0)} = e^{-\alpha z}
\]

1 km \( \Rightarrow \frac{V(z=1 \text{ km})}{V(z=0)} = e^{-\alpha \times 1000} = 0.317 \), \( 31.7\% \).
\[ f = 500 \text{ MHz}, \quad Z_0 = 80 \Omega, \quad \alpha = 0.04 \text{ Np/m} \]
\[ \beta = 1.5 \text{ rad/m} \quad \text{Find } R, L, G, \text{ & } C \quad ?? \]

\[ \gamma = \sqrt{Z Y} = \sqrt{(R + j\omega L)(G + j\omega C)} \]

\[ Z_0 = \sqrt{\frac{Z}{Y}} \quad \Rightarrow \quad \frac{1}{\gamma} = \frac{1}{Z_0} \quad \Rightarrow \quad \sqrt{Z} = Z_0 \sqrt{Y} \]

\[ \therefore \quad \gamma = \frac{Z}{Z_0} \quad \& \quad \gamma = Z_0 \gamma \]

\[ (\alpha + j\beta) = \frac{R + j\omega L}{80} \]

\[ R = 3.2 \quad \Omega/m \]

\[ L = 38.2 \quad \text{nH/m} \]

\[ \alpha + j\beta = 80 (G + j\omega C) \]

\[ \Downarrow \quad G = 5 \times 10^{-4} \quad \text{S/m} \]

\[ C = 5.97 \quad \text{pF/m} \]

\[ \text{OR} \quad \text{Since } Z_0 \text{ is real} \quad \& \quad \alpha \neq 0 \]

\[ \Rightarrow \text{ distortionless line} \]

\[ Z_0 \beta = \omega L \quad \Rightarrow \quad L = 38.2 \quad \text{nH/m} \]

\[ Z_0 / \beta = \frac{1}{\omega C} \quad \Rightarrow \quad C = 5.97 \quad \text{pF/m} \]

\[ \alpha = R \sqrt{\frac{C}{L}} \quad \Rightarrow \quad R = \alpha Z_0 = 3.2 \quad \Omega/m \]

\[ G = C \frac{R}{L} = 5 \times 10^{-4} \quad \text{S/m} \]
consider,

\[ V(z) = V_o^+ e^{-|z|} + V_o^- e^{\gamma |z|} \]

\[ I(z) = \frac{V_o^+}{Z_o} e^{-\gamma |z|} - \frac{V_o^-}{Z_o} e^{\gamma |z|} \]

To find \( V_o^+ \) & \( V_o^- \), terminal conditions must be given.

1. If given
   \[ V_o = V(z=0) \] & \( I_o = I(z=0) \)

   \[ \Rightarrow V_o^+ = \frac{1}{2} \left( V_o + Z_o I_o \right) \]
   \[ V_o^- = \frac{1}{2} \left( V_o - Z_o I_o \right) \]
   
   note: \[ V_o = V_g \frac{Z_{in}}{Z_{in} + Z_g} \], \[ I_o = \frac{V_g}{Z_{in} + Z_g} \]

2. If given
   \[ V_L = V(z=l) \] & \( I_L = I(z=l) \)

   \[ \Rightarrow V_o^+ = \frac{1}{2} \left( V_L + Z_o I_L \right) e^{\gamma l} \]
   \[ V_o^- = \frac{1}{2} \left( V_L - Z_o I_L \right) e^{-\gamma l} \]
Note:

$$V_L = I_L Z_L$$

$$V_o^+ = \frac{1}{2} I_L (Z_L + Z_o) \hat{e}$$

$$V_o^- = \frac{1}{2} I_L (Z_L - Z_o) \hat{e}$$

Substituting in $V(\hat{e})$ & $I(\hat{e})$ and using the facts

$$\hat{e}^\theta + \hat{e}^{-\theta} = 2 \cosh \theta \quad \& \quad \hat{e}^\theta - \hat{e}^{-\theta} = 2 \sinh \theta$$

gives

$$V(\hat{e}) = I_L \left[ Z_L \cosh \gamma \hat{e} + Z_o \sinh \gamma \hat{e}' \right]$$

$$I(\hat{e}) = \frac{I_L}{Z_o} \left[ Z_L \sinh \gamma \hat{e} + Z_o \cosh \gamma \hat{e}' \right]$$

Where $\hat{e}' = l - z$ (distance from load. Book calls it $\hat{e}'$)

\[ l' = l - z \]

- Input impedance at a distance $l'$ from the load

$$Z(l') = \frac{V(l')}{I(l')} = Z_o \frac{Z_L + Z_o \tanh \gamma l'}{Z_O + Z_L \tanh \gamma l'}$$
input impedance at generation end, i.e., at 
\( z = 0 \) or \( l = l' \)

\[
Z_{in} = Z(\ l' = 0) = Z_0 \ \frac{Z_L + Z_0 \ \text{tanh} \ \gamma l}{Z_0 + Z_L \ \text{tanh} \ \gamma l}
\]

\[
= \ \frac{V(z=0)}{I(z=0)} = \ \frac{V_0}{I_0} = Z_0 \ \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}
\]

For a lossless line, \( \gamma = j \beta \Rightarrow \ \text{tanh} \ \gamma l = \ \text{tanh} \ j \beta l \)

\[
\Rightarrow Z(l) = Z_0 \ \frac{Z_L + j Z_0 \ \text{tan} \ \beta l}{Z_0 + j Z_L \ \text{tan} \ \beta l'}
\]

Define: electrical length = \( \beta l = \frac{\alpha}{\lambda} l \) (rad or degrees)

Define: voltage reflection coefficient

\[
\Gamma(z) = \ \frac{V_0^- \ e^{\sigma z}}{V_0^+ \ e^{\sigma z}} = \ \frac{V_0^-}{V_0^+} \ \frac{e^{\sigma z}}{e^{\sigma z}} = \ \frac{\lambda}{\lambda} \ \frac{(V_L - I L Z_0) \ e^{\sigma z}}{(V_L + I L Z_0) \ e^{\sigma z}} = \ \frac{Z_L - Z_0}{Z_L + Z_0} \ \frac{e^{2 \sigma z}}{e^{2 \sigma z}}
\]

\[
\Gamma(z) = \ \frac{Z_L - Z_0}{Z_L + Z_0} \ \frac{e}{e} \ \frac{2 \sigma z}{2 \sigma z}
\]

\[
\Gamma(z) = \ \frac{Z_L - Z_0}{Z_L + Z_0} \ \frac{-2 \sigma z}{-2 \sigma z}
\]
At load end, $z=l$, $l'=0$

$$\Rightarrow \Gamma(z=l) = \frac{Z_l-Z_0}{Z_l+Z_0} = |P_L| e^{j\theta_P}$$

\[ \therefore \Gamma(z) = P_L e^{-2\sigma l'} \]

- **Note:** for a lossless line, $\gamma = j\beta$

$$\Gamma(z) = P_L e^{-j2\beta l'}$$

$$|\Gamma(z)| = |P_L|$$

\[ \therefore |P_L| \text{ is constant along a lossless line, and is equal to } |P_L| \]

- **Note:** current reflection coeff. $= \frac{I_o^- e^{-\gamma z}}{I_o^+ e^{-\gamma z}}$

$$= \frac{-V_o/Z_0}{V_o^+/Z_0} e^{2\gamma z}$$

$$= -\Gamma(z)$$

- $\Rightarrow V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$

$$= V_o^+ e^{-\gamma z} [1 + \Gamma(z)]$$

$$= V_o^+ e^{-\gamma z} [1 + P_L e^{-2\sigma l'}]$$
\[ I(\xi) = \frac{V_0^+}{Z_o} e^{-r_0} \left[ 1 - \frac{P_L}{1 - P_L} e^{-2r_2} \right] \]

**Note:**
\[
Z(\xi) = \frac{V(\xi)}{I(\xi)} = Z_o \left( \frac{1 + P(\xi)}{1 - P(\xi)} \right)
\]
\[
P(\xi) = \frac{Z(\xi) - Z_o}{Z(\xi) + Z_o}
\]
\[
P(\xi = l) = R_L = \frac{Z_L - Z_o}{Z_L + Z_o}
\]

- If \( Z_L = Z_o \Rightarrow R_L = \text{zero} \Rightarrow P(\xi) = \text{zero} \) (matched load)

**Lossless Line:**
\[
V(\xi) = V_o^+ e^{j\beta_2} \left[ 1 + P_L e^{j\alpha_2} \right]
\]

represents a standing wave pattern

\[
\text{define: } \text{SWR} = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{I_{\text{max}}}{I_{\text{min}}} = \frac{1 + |P|}{1 - |P|} = \frac{1 + |P|}{1 - |P|}
\]

- \( V(\xi) = V_o^+ e^{j\beta_2} \left[ 1 + |P| e^{j\alpha_2} e^{-j2\alpha_L} \right] \)

- \( I(\xi) = \frac{V_o^+}{Z_o} e^{j\beta_2} \left[ 1 - |P| e^{j\alpha_L} e^{-j2\alpha_L} \right] \)
\( V_{\text{max}} \) & \( I_{\text{min}} \) occur together when
\[
\frac{\theta_{p_c}}{2\beta} \frac{d}{d\theta_{p_c}} = -2n\pi, \quad n = 0, 1, 2, \ldots
\]

\[
\Rightarrow \frac{d}{d\theta_{p_c}} = \frac{\theta_{p_c} + 2n\pi}{2\beta}
\]

\[
V_{\text{max}} = \frac{1}{V_{0}^{+}} \left( 1 + 1\pi \right)
\]

\[
I_{\text{min}} = \frac{V_{0}^{+}}{Z_{0}} \left( 1 - 1\pi \right)
\]

The max. input imp. along the line

\[
Z_{\text{max}} = \frac{V_{\text{max}}}{I_{\text{min}}} = Z_{0} \frac{1 + 1\pi}{1 - 1\pi} = Z_{0}s \quad \text{(real quantity)}
\]

\( V_{\text{min}} \) & \( I_{\text{max}} \) occur together when
\[
\frac{\theta_{p_c}}{2\beta} \frac{d}{d\theta_{p_c}} = -(2n + 1)\pi, \quad n = 0, 1, 2, \ldots
\]

\[
\Rightarrow \frac{d}{d\theta_{p_c}} = \frac{\theta_{p_c} + (2n + 1)\pi}{2\beta}
\]

\[
V_{\text{min}} = \frac{1}{V_{0}^{+}} \left( 1 - 1\pi \right)
\]

\[
I_{\text{max}} = \frac{V_{0}^{+}}{Z_{0}} \left( 1 + 1\pi \right)
\]

\[
Z_{\text{min}} = \frac{V_{\text{min}}}{I_{\text{max}}} = \frac{Z_{0}}{s} \quad \text{(real quantity)}
\]

Distance between successive maxima (or successive minima)
\[
d = \left. \frac{d}{d\theta_{p_c}} \right|_{n=1} - \left. \frac{d}{d\theta_{p_c}} \right|_{n=0} = \frac{\theta_{p_c} + 2\pi}{2\beta} - \frac{\theta_{p_c}}{2\beta} = \frac{\pi}{\beta} = \frac{\lambda}{2}
\]
Conditions on the line repeat themselves every half wavelength.

Connecting an oscilloscope at any point along the line will give a sine wave as a function of time.

Connecting an ac voltmeter along the line will give the standing wave pattern.
Special Cases:

1. Open-Circuited Line \((Z_L = \infty)\)

\[ P_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 1 \rightarrow |P| = 1, \quad \Theta_L = \text{zero}, \quad \text{swk} = \infty \]

\[ |V(z)| = |V_o^+| \left| 1 + |P_L| e^{j(\Theta_L - 2\beta z')} \right| \]

\[ = |V_o^+| \left| 1 + e^{-j2\beta z'} \right| \]

\[ = |V_o^+| \sqrt{(1 + \cos 2\beta z')^2 + \sin^2(2\beta z')} \]

\[ = |V_o^+| \sqrt{2 + 2 \cos(2\beta z')} \]

\[ \approx \sqrt{2} |V_o^+| \sqrt{1 + \cos(2\beta z')} \]

\[ = \sqrt{2} |V_o^+| \sqrt{2 \cos^2(\beta z')} \]

\[ = 2 |V_o^+| \left| \cos(\beta z') \right| = |V_L| \left| \cos(\beta z') \right| \]

\[ |I(z)| = \frac{a |V_o^+|}{Z_0} |\sin \beta z'| \]
\[ Z(z) = Z_o \frac{Z_L + jZ \tan \beta l'}{Z_o + jZ_L \tan \beta l'} \]

\[ = \frac{Z_o}{j \tan \beta l'} = -j \frac{Z_o}{\cot (\beta l')} \]

\( Z(z) \) is pure reactive, could be capacitive or inductive depending on \( \beta l' \).

In practice, an ideal open is not possible.

(c) Short-Circuited Line \( (Z_L = \text{zero}) \)

\[ \rho_L = \frac{Z_L - Z_o}{Z_L + Z_o} = -1 \rightarrow |\rho_L| = |\rho| = 1, \quad \theta_{\rho_L} = \pi \]

\( \text{swr} = \infty \)

\[ |V(z)| = |I_L|Z_o |\sin (\beta l')| \]

\[ |I_L(z)| = |I_L| |\cos (\beta l')| \]

\[ Z(z) = j Z_o \tan (\beta l') \]

\[ Z_{oc} = -j Z_o \cot (\beta l'), \quad Z_{sc} = j Z_o \tan (\beta l') \]

\[ Z_{oc} Z_{sc} = Z_o^2 \quad \& \quad \tan (\beta l') = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \]
(3) Matched load, \( Z_L = Z_0 \)

\[
\eta_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \text{zero}
\]

\[
\text{SWR} = \frac{1 + |\eta|}{1 - |\eta|} = 1
\]

\[
|V(z)| = |V_0^+|
\]

\[
V(z) = V_0^+ e^{-j\beta z}, \quad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}
\]

No standing wave

No reflected wave

as if the line is of infinite length

\[ Z(z) = Z_0 \quad (\text{input impedance at any point along the line} = Z_0) \]

All of incident power is delivered to the load. Max. power transfer to the load.

(4) Resistive Load, \( Z_L = R_L \)

\[
\eta_L = \frac{R_L - R_0}{R_L + R_0} \quad \text{real number}
\]

(a) \( R_L > R_0 \Rightarrow \eta_L > 0 \Rightarrow \theta_{\eta_L} = 0 \)

\[ V_{\text{max}} \ & \ I_{\text{min}} \ \text{occur when} \]

\[ \theta_{\eta_L} - 2\beta z_{\text{max}} = -2n\pi \]

\[ 2\beta z_{\text{max}} = 2n\pi \Rightarrow z_{\text{max}} = \frac{n\lambda}{2}, \quad n=0,1,2,... \]
\[\text{SWR} = \frac{V_{\max}}{V_{\min}} = \frac{1 + r_L}{1 - r_L} = \frac{R_L}{R_0}\]

(b) \(R_L < R_0 \Rightarrow r_L < 0 \Rightarrow \theta_L = \pi\)

In this case, it can be shown that \(V_{\min}\) & \(I_{\max}\) occur at \(\hat{z}_{\min} = \frac{n\lambda}{2}, n = 0, 1, \ldots\)
Pure reactive load, \( Z_L = jX_L \)

\[
P_L = \frac{jX_L - R_o}{jX_L + R_o}
\]

\(|P_L| = 1 \Rightarrow \text{SWR} = \infty\]

\[
\theta_{P_L} = \tan^{-1}\left( \frac{X_L}{-R_o} \right) - \tan^{-1}\left( \frac{X_L}{R_o} \right)
\]

\[
= \frac{\pi}{2} + \tan^{-1}\left( \frac{R_o}{X_L} \right) - \frac{\pi}{2} + \tan^{-1}\left( \frac{R_o}{X_L} \right)
\]

\[
= 2 \tan^{-1}\left( \frac{R_o}{X_L} \right)
\]

\[
|V(z)| = |V_0^+| \left| 1 + e^{j\theta_{P_L}} e^{-j2\beta z'} \right|
\]

\[
|V(z)| = 2 |V_0^+| \left| \cos \left( \beta z' - \frac{\theta_{P_L}}{2} \right) \right|
\]

(a) \( X_L > 0 \), inductive

\[
\Rightarrow 0 < \theta_{P_L} < \pi
\]

\[
V_{\text{max}}: \quad 2\beta z'_{\text{max}} = \theta_{P_L} + 2n\pi \quad n = 0, 1, 2, \ldots
\]

\[
V_{\text{min}}: \quad 2\beta z'_{\text{min}} = \theta_{P_L} + n\pi \quad n = 1, 3, 5, \ldots
\]
(b) \( X_L < 0 \), capacitive

\[ \pi < \Theta_{pL} < 2\pi \]

\[
\begin{align*}
|V| & \quad \text{vs.} \quad \Theta_{pL}
\end{align*}
\]

(6) Quarter wave section, \( l = (2n+1) \frac{\lambda}{4} \), \( n = 0, 1, 2, \ldots \)

\[
\beta l = \frac{2\pi}{\lambda} \left(2n+1\right) \frac{\lambda}{4} = \left(2n+1\right) \frac{\pi}{2}
\]

\[ \Rightarrow \tan \beta l \rightarrow \pm \infty \]

\[ Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \]

\[ \therefore Z_{in} = \frac{Z_0^2}{Z_L} \]

If \( Z_L = \infty \) (o.c.) \( \Rightarrow \) \( Z_{in} = \text{zero} \)

\( Z_L = 0 \) (s.c.) \( \Rightarrow \) \( Z_{in} = \infty \)

(7) Half wave section, \( l = n \frac{\lambda}{2} \), \( n = 1, 2, \ldots \)

\[
\beta l = \frac{2\pi}{\lambda} n \frac{\lambda}{2} = n \pi
\]

\[ \Rightarrow \tan \beta l = \text{zero} \Rightarrow Z_{in} = Z_L \]
\[ \omega = 10^6 \text{ rad/s} \]
\[ \alpha = 8 \, \text{dB/m} = \frac{8}{8.69} = 0.921 \, \text{Np/m} \]
\[ \beta = 1 \, \text{rad/m} \]
\[ Z_0 = 60 + j 40 \quad , \quad l = 2 \, \text{m} \]
\[ V_g = 10 \angle 0^\circ \quad , \quad Z_g = 40 \, \Omega \quad , \quad Z_L = 20 + j 50 \, \Omega \]

(1) Find \( Z_{in} \).
\[ Z_{in} = Z_0 \left( \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right) \]
\[ \gamma = \alpha + j \beta = 0.921 + j 1 \]
\[ \tanh (x+jy) = \frac{\sinh (x+jy)}{\cosh (x+jy)} = \frac{\sinh(x) \cosh(jy) + \cosh(x) \sinh(jy)}{\cosh(x) \cosh(jy) + \sinh(x) \sinh(jy)} \]
\[ = \frac{\sinh(x) \cos(y) + j \cosh(x) \sin(y)}{\cosh(x) \cos(y) + j \sinh(x) \sin(y)} \]
\[ Z_{in} = 60.25 + j 38.8 \] \( \text{\Omega} \)

(2) \( I(\zeta=0) = I_o \) = ?
\[ I_o = \frac{V_g}{Z_g + Z_{in}} = 93.03 \angle -21.15^\circ \]

(3) \( I(\zeta) = ? \)
\[ I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} = \frac{V_0^-}{Z_0} e^{\gamma z} \]

\[ V_o^+ = \frac{1}{z} \left( V_o + Z_0 I_o \right) = 6.687 \angle 121^\circ \]
\[ V_o^- = \frac{1}{z} \left( V_o - Z_0 I_o \right) = 0.0518 \angle 260^\circ \]

where \( V_o = I_o \cdot Z_{in} \)

\text{P. E. 11.3}

\( \ell = 40 \text{ m} \), \( V_g = 15 \angle 70^\circ \), \( Z_o = 30 + j60 \)
\( V_L = 5 \angle -48^\circ \). Line is matched to the load.
\( Z_g = Z_o \)

\( \cdot \text{Since matched load } \Rightarrow Z_L = Z_o = 30 + j60 \)

\( Z_{in} = ? \)

Since \( Z_L = Z_o \) \( \Rightarrow \) \( Z(z) = Z_o \)
\( \Rightarrow Z_{in} = Z(z=0) = Z_o \)

\( \cdot I_{in} = ? \), \( V_{in} = ? \)
\[ I_{in} = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15}{2(30+j60)} \]
\[ V_{in} = Z_{in} I_{in} = Z_o \frac{V_g}{2Z_o} = \frac{V_g}{2} \]

\( \gamma = \alpha + j\beta = ?? \)
\[ V(z) = V_o^+ e^{-\gamma z} \]
\[ V(z=0) = V_o^+ = V_{in} \Rightarrow V_o^+ = V_{in} \]
\[ V(z=\ell) = V_L = V_{in} e^{-\alpha \ell - j\beta \ell} \]}
\[ 5 \cdot 98^\circ = 7.5 \cdot e^{\alpha l} \cdot e^{j\beta l} \]

\[ 5 = 7.5 \cdot e^{\alpha l} \Rightarrow \alpha = 0.0101 \quad \text{N/m} \]

\[ \frac{48^\circ}{180^\circ} \times \pi = \beta \cdot l \Rightarrow \beta = 0.02094 \quad \text{rad/m} \]

---

**Ex.**

\[ l = 100 \text{ m} \quad \text{,} \quad f = 3 \times 10^9 \text{ Hz} \quad \text{,} \quad \text{air-filled TL} \]

\[ Z_0 = 50 \Omega \quad \text{,} \quad Z_g = 2 \cdot Z_0 \quad \text{,} \quad V(x=0) = V_i = 5 \text{ (V)} \]

Find \( Z_l \) !

\[ V_i = V_g \cdot \frac{Z_i}{Z_i + Z_g} \Rightarrow Z_i = Z_g = 2 \cdot Z_0 \]

\[ \lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^9} = 0.1 \quad \text{m} \]

\[ \therefore l = 100 \text{ m} = 1000 \lambda \]

i.e. \( l \) is an integer multiple of \( \lambda \)

\[ \therefore Z_l = Z_{in} = 2 \cdot Z_0 \]
\[ P_{\text{av}}(z) = \frac{1}{a} \text{Re} \left\{ V(z) I^*(z) \right\} \]

\[ P_{\text{av}}(z = L) = \frac{1}{a} \text{Re} \left\{ V_L I_L^* \right\} \]

\[ = \frac{1}{a} \frac{V_L}{Z_L} I_L^2 R_L = \frac{1}{a} \left| I_L \right|^2 P_L \]

- For a lossless line,

\[ P_{\text{av}}(z) = \text{const.} = (P_{\text{av}})_{\text{load}} \]

\[ V(z) = V_o^+ e^{-j \beta z} \left( 1 + \Gamma(z) \right)^+, \quad \Gamma(z) = \frac{V_o^-}{V_o^+} e^{j 2 \beta z} \]

\[ = V_o^+ e^{-j \beta z} + V_o^+ \Gamma(z) e^{j \beta z} \]

\[ = V_{\text{inc}} + V_{\text{ref}} \]

\[ I(z) = \frac{V_o^+}{Z_o} e^{-j \beta z} \left( 1 - \Gamma(z) \right)^- \]

\[ = \frac{V_o^+}{Z_o} e^{-j \beta z} - \frac{V_o^+}{Z_o} \Gamma(z) e^{j \beta z} \]

\[ = I_{\text{inc}} + I_{\text{ref}} \]

\[ P_{\text{inc}} = \frac{1}{a} \text{Re} \left\{ V_{\text{inc}} I_{\text{inc}}^* \right\} = \frac{1}{a} \frac{\left| V_o^+ \right|^2}{Z_o} \]

\[ P_{\text{ref}} = \frac{1}{a} \text{Re} \left\{ V_{\text{ref}} I_{\text{ref}}^* \right\} = \frac{1}{a} \frac{\left| V_o^- \right|^2}{Z_o} \left| \Gamma \right|^2 = \left| \Gamma \right|^2 P_{\text{inc}}. \]

\[ P_{\text{load}} = P_{\text{inc}} - P_{\text{ref}} = (1 - \left| \Gamma \right|^2) \frac{\left| V_o^+ \right|^2}{2 Z_o} \]

(also, \( P_{\text{ref}} = \frac{1}{2} \frac{\left| V_o^- \right|^2}{Z_o} \))
Ex.

\[ Z_0 = 50 \text{ ohms} \]

Power delivered to load = 1 K watt

\[ V_{\text{max}} \text{ along the line} = 250 \text{ rms volts} \]

Find maximum allowable SWR

Find \( P_{\text{incident}} \)

\[ V_{\text{max}} = |V_0^+| (1 + 1\|l\|) \]

\[ \Rightarrow \ |V_0^+| (1 + 1\|l\|) \leq (250) \sqrt{2} \]

\[ P_L = \frac{|V_0^+|^2}{2Z_0} (1 - 1\|l\|^2) = 1000 \]

\[ \Rightarrow \ |V_0^+|^2 = \frac{2000 Z_0}{1 - 1\|l\|^2} \]

\[ \Rightarrow \ \frac{2000 Z_0}{1 - 1\|l\|^2} (1 + 1\|l\|^2) \leq (250)^2(2) \]

\[ \frac{1 + 1\|l\|}{1 - 1\|l\|} \leq \frac{(250)^2(2)}{(2000)(50)} = 1.25 \]

\[ \Rightarrow \text{SWR} \leq 1.25 \]

\[ P_L = P_{\text{inc}} (1 - 1\|l\|^2) \]

\[ = P_{\text{inc}} \left[ 1 - \left( \frac{S-1}{S+1} \right)^2 \right] = P_{\text{inc}} \left[ \frac{4S}{(S+1)^2} \right] \]

\[ \Rightarrow P_{\text{inc}} = 1012.5 \text{ watts} \]
\[ Z_g = 5Z_0 \]

\[ V(\frac{\ell}{2} = 0) = V_{\text{max}} = 5 \text{ volts} \]

Find \( V_g \) ??

Since \( Z_L \) is pure real & \( Z_L < Z_0 \), the following standing wave will exist on the line

\[ V_{\text{max}} = 5 \text{ volts} \]

\[ \|V\| \]

: The length of the line \( \ell \) is an odd multiple of \( \frac{\lambda}{4} \).

\[ Z_{\text{in}} = \frac{Z_0^2}{Z_L} = 5Z_0 \]

\[ V_{\text{in}} = V_g \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_g} \Rightarrow 5 = V_g \frac{5Z_0}{5Z_0 + 5Z_0} \]

\[ \therefore \ V_g = 10 \]
Example:

\[ Z_0 = 60 \Omega \quad \overrightarrow{P_1} \quad \overrightarrow{P_2} \quad Z_{02} \]

Lossless lines

\[ P_1 = 12 \text{ mW}, \quad P_2 = 9 \text{ mW}, \quad \text{find } Z_{02}. \]

\[ P_{\text{ref}} = 12 - 9 = 3 \text{ mW} \]

\[ |P| = \frac{P_{\text{ref}}}{P_{\text{inc}}} = \frac{3}{12} = \frac{1}{4} \quad \Rightarrow \quad |P| = 0.5 \]

\[ \text{SWR} = \frac{1 + |P|}{1 - |P|} = 3 \]

Since lossless T.L.'s \( \Rightarrow \) \( Z_{02} \) is real

Since \( V_{\text{min}} \) occurs at the load

\( \Rightarrow \) \( Z_{02} \ll Z_{01} \) \( \Rightarrow \) \( P < 0 \)

\( \Rightarrow \) \( \text{SWR} = \frac{Z_{01}}{Z_{02}} \Rightarrow Z_{02} = \frac{60}{3} = 20 \ \Omega \)

Note:

\[ |V_{0+}^1| \neq |V_{0+}^2| \]
11.5 The Smith Chart

- Smith chart is a graphical solution for T.L. problems.
- We'll assume lossless line \((Z_0 = R_0)\) although it can be used for lossy lines too.

\[
\Pi_L = \frac{Z_L - R_0}{Z_L + R_0} = |\Pi_L| e^{j\theta_L}
\]

let \(z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j \frac{X_L}{R_0} = r + jx\)

\[\Rightarrow \Pi_L = \frac{z_L - 1}{z_L + 1} = \Pi_r + j\Pi_i \quad \text{imag. part}
\]

\[\Rightarrow z_L = \frac{1 + \Pi_L}{1 - \Pi_L} \Rightarrow r + jx = \frac{(1 + \Pi_r) + j\Pi_i}{(1 - \Pi_r) - j\Pi_i}
\]

\[\Rightarrow r = \frac{1 - \Pi_r^2 - \Pi_i^2}{(1 - \Pi_r)^2 + \Pi_i^2} \quad \text{&} \quad x = \frac{2\Pi_i}{(1 - \Pi_r)^2 + \Pi_i^2}
\]

1. \((\Pi_r - \frac{r}{1 + r})^2 + \Pi_i^2 = (\frac{1}{1 + r})^2\)
2. \((\Pi_r - 1)^2 + (\Pi_i - \frac{1}{x})^2 = (\frac{1}{x})^2\)
1. Is the eq. of a circle \((r-\text{circle})\)
   center at \((P_r, P_i) = \left(\frac{r}{1+r}, 0\right)\)
   radius = \(\frac{1}{1+r}\)

2. Is the eq. of a circle \((x-\text{circle})\)
   center at \((P_r, P_i) = \left(1, \frac{1}{x}\right)\)
   radius = \(\frac{1}{x}\)

<table>
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<tr>
<th>(r)</th>
<th>(r_\text{radius})</th>
<th>(r_\text{center})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>((0,0))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td>((\frac{1}{3}, 0))</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{1}{1})</td>
<td>((\frac{1}{2}, 0))</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{1}{3})</td>
<td>((\frac{2}{3}, 0))</td>
</tr>
<tr>
<td>(\infty)</td>
<td>0</td>
<td>((1,0))</td>
</tr>
</tbody>
</table>

- Centers lie on \(P_r\) axis.
- All \(r\)-circles pass through \((1, 0)\).
- Largest circle is when \(r = 0\) (the unity circle).
- \(r = \infty\) is the open-circuit \(\Rightarrow\) a single point.
- Circles become smaller as \(r\) increases.
<table>
<thead>
<tr>
<th>$x$</th>
<th>radius</th>
<th>center</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\infty$</td>
<td>$(1, \infty)$</td>
</tr>
<tr>
<td>$\pm \frac{1}{2}$</td>
<td>2</td>
<td>$(1, \pm 2)$</td>
</tr>
<tr>
<td>$\pm 1$</td>
<td>1</td>
<td>$(1, \pm 1)$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>$\frac{1}{2}$</td>
<td>$(1, \pm \frac{1}{2})$</td>
</tr>
<tr>
<td>$\pm \infty$</td>
<td>0</td>
<td>$(1, 0)$</td>
</tr>
</tbody>
</table>

- Centers lie on $P_r = 1$ line.
- Circles become smaller as $x$ increases.
- All circles pass through $(1, 0)$.
- $x = 0$ corresponds to a pure resistive impedance ($Z_L = R_L$) $\Rightarrow$ $P_r = \frac{R_L - R_0}{R_L + R_0}$ real number $\Rightarrow$ lies on the $P_r$-axis.
- $x = \pm \infty$ is the open circuit $\Rightarrow$ open circuit.
- Intersection of an $r$-circle with an $x$-circle gives the normalized impedance $r + jx$. 
\[ S \text{-} \text{circles (constant standing-wave-ratio circles)} \]
are circles centered at the origin with \( S \) varying from 1 to \( \infty \).

- Called 1\( |P| \)-circles also since they represent constant 1\( |P| \) too, varying from zero to 1.
- \( r = 0 \) circle corresponds to 1\( |P| = 1 \) circle or \( S = \infty \) circle (pure reactive load).
- \((r = 1 \& \varphi = 0)\) point corresponds to \( r = \text{zero} \Rightarrow S = 1 \) (matched load).

- Locate \( P_o \& P_{sc} \) on Smith chart.

- As you move along a T.L., one moves on a constant 1\( |P| \)-circle on Smith Chart since 1\( |P| \) const. on a lossless line.

- Moving clockwise on 1\( |P| \) circle is regarded as moving towards generator along the T.L.
  
  \[ P(z) = P_L e^{-j \beta z} \]
  
  \[ \Theta_{P(z)} = \Theta_P - 2 \beta z \]

- Moving counterclockwise is regarded as moving towards load along the T.L.

- Discuss scales & how to find 1\( |P| \).
$P_m$ & $P_M$ represent pure resistances

at $P_m$ : $R_L > R_0$. But, $S = \frac{R_L}{R_0} = r$

: the value of the $r$-circle passing through
$P_m$ is $= S$

at $P_m$ : $R_L < R_0$. But, $S = \frac{R_0}{R_L} = \frac{1}{r}$

: the value of the $r$-circle passing through
$P_m$ is $= 1/S$

Also, recall that along the T.I.L.

$Z_{max}$ is pure real and equals $S \cdot R_0$

$Z_{min}$ = \dots \dots \dots \frac{R_0}{S}$

= at $P_M$ : $Z_{max}$, $V_{max}$ & $I_{min}$ are located

at $P_m$ : $Z_{min}$, $V_{min}$ & $I_{max}$
* A complete revolution around Smith Chart represents a distance \( \lambda/2 \) on the line.

\[
\text{phase of } \pi = \Theta_p - 2\beta \gamma
\]

Let \( \gamma = \lambda/2 \) \( \Rightarrow \) phase = \( \Theta_p - 2 \frac{2\pi}{\lambda} \frac{1}{2} \)

\[
= \Theta_p - 2\pi
\]

(complete revolution)

* \( Z(\gamma) \) repeats every \( \lambda/2 \).

* Smith chart can be used as an admittance chart

\( r- \text{circle} \rightarrow g- \text{circle} \)

\( x- \rightarrow b- \)

Let \( Y_L = \frac{1}{Z_L} \) load admittance

\( Z_L = \frac{Z_L}{R_o} = \frac{1}{R_o Y_L} = \frac{1}{Y_L} \)

\( Y_L = R_o Y_L = \frac{Y_L}{Y_o} = \frac{Y_L}{G_o} = g + j b \)

(normalized admittance)

\( \lambda/\gamma \)

\( Z_o \)

\( \lambda/\gamma \)

\( Z_L \)

\( Z_L \)

\( Z_{in} = \frac{Z_o^2}{Z_L} \Rightarrow \frac{Z_{in}}{Z_o} = \frac{Z_o}{Z_L} = \frac{1}{Z_L} = Y_L \)

\( \Rightarrow \frac{Z_{in}}{Z_o} = Y_L \)
moving along a 180 circle a distance of 1/4 gives the admittance.

at $y_L$ read the values of the $g$-circle & $b$-circle $\Rightarrow y_L = g + jb$

Note: you still read positive $b$ in the upper region, & negative $b$ in the lower region.

Note: $b > 0$ corresponds to capacitive load; $b < 0$ , inductive load.
Ex. 11.4

\[ l = 30 \text{ m}, \text{ lossless T.L.} \]
\[ Z_0 = 50 \Omega, \quad f = 2 \text{ MHz} \]
\[ Z_L = 60 + j40, \quad \mu = 0.6 \text{ c} \]

find \( R_L \), SWR, \( Z_m \)?

1. Read book for analytical solution.

2. Using Smith Chart:

\[ z_L = \frac{Z_L}{Z_0} = \frac{60 + j40}{50} = 1.2 + j0.8 \]

3. Locate \( z_L \) on Smith chart as the intersection of \( r = 1.2 \) circle & \( x = 0.8 \) circle. (point P)

4. Extend OP to meet \( r=0 \) circle at Q.

5. Measure \( \overline{OP} \) & \( \overline{OQ} \):

\[ 1P_L = \frac{\overline{OP}}{\overline{OQ}} \approx 0.352 \]

Or use lower scale for \( 1P_L \).

6. \( \overline{OQ} \) corresponds to \( 1P_L = 1 \).
read $\Omega_R$ directly from Smith Chart

the angle that $OP$ makes with the positive $P_y$ axis.

$\Omega_R = 56^\circ$

$P_L = 0.352 \angle 56^\circ$

to get SWR, can use lower scale for SWR directly. or draw 181 circle, the value of the $r$-circle at $P_M$ equals SWR

$\Rightarrow$ SWR = 2.1

to get $Z_{in}$, first express $\lambda$ in wavelengths.

$\lambda = \frac{u}{f} = \frac{(0.6)(3 \times 10^8)}{2 \times 6} = 90 \text{ m}$

$\lambda = 30 \text{ m} = \frac{30}{90} \lambda = \frac{1}{3} \lambda = 0.333 \lambda$

Extend the line $OQ$, observe its reading using the "wavelengths towards generator" scale

reading = 0.172 $\lambda$  -- this is the location of $Z_L$ on the Smith Chart.

Now, we need to move a distance of 0.333 $\lambda$

T.G. to find $Z_{in}$

$\Rightarrow (0.172 + 0.333)\lambda = 0.505 \lambda$

locate 0.505 $\lambda$ on T.G. scale, connect it with the origin, notice the intersection with the 181- circle, point K.
at point $K$, read the impedance

$$z_m = 0.47 + j 0.035$$

$$Z_m = z_m Z_o = 23.5 + j 1.75$$

Ex. 11.5

$Z_L = 100 + j 150$, $Z_o = 75 \Omega$, $l = 0.64$

find $\Pi_L$, $SWR$, $Y_L$, $Z(\varepsilon = 0.41)$, locations of $V_{max}$ & $V_{min}$, $Z_m$ !!

- $z_L = \frac{Z_L}{Z_o} = 1.33 + j 2$

- locate $z_L$ on Smith Chart $\rightarrow$ point $P$

- $\Pi_L = \frac{OP}{OQ} = 0.66$, $\Theta_L = 40^\circ$

- $V_L = 0.66 \angle 40^\circ$

- $SWR = 4.82$ (use lower scale)
To get \( Y_L \), extend PO to \( P' \).
read the value of \( Y_L \) at \( P' \)
\[
Y_L \approx 0.228 - j 0.35
\]
\[
Y_L = Y_0 Y_L = 3.04 - j 4.67 \text{ (mS)}
\]

To get \( Z(\varepsilon = 0.4 \lambda) \), locate extend OA & observe its intersection with T.G. scale
\[
\varepsilon = 0.194 \lambda
\]
Now, \((0.194 + 0.4) = 0.594 \lambda\)
subtract \(0.5 \lambda \rightarrow 0.094 \lambda\)
locate \(0.094 \lambda\) on T.G. scale
connect it to the origin & read its intersection with the 171-circle (point K)
read at point K:
\[
Z(\varepsilon = 0.4 \lambda) = (0.3 + j 0.63)(Z_0) = (22.5 + j 47.25) \Omega
\]

Similarly, for \( Z_{in} \):
\[
(0.194 + 0.6) = 0.794 \lambda
\]
\[
(0.794 - 0.5) = 0.294 \lambda \rightarrow \text{point N}
\]

\[
Z_{in} = (1.8 - j 2.2) Z_0 = (135 - j 165) \Omega
\]

\( V_{max} \) occurs at \( P_{max} \):
\[
\left( \frac{V_{max}}{I} \right)_{1} = (0.25 - 0.144) \lambda = 0.056 \lambda \text{ from load}
\]
\[
\left( \frac{V_{max}}{I} \right)_{2} = (0.056 + 0.5) \lambda = 0.556 \lambda
\]
\[
\left( \frac{V_{min}}{I} \right)_{1} = (0.056 + 0.25) \lambda = 0.306 \lambda
\]
**Ex:** \( Z_0 = 50 \Omega, \quad \text{SWR} = 3, \quad \lambda = 0.4 \text{ m} \)

First \( V_{\text{min}} \) at \( \hat{z}_{\text{min}} = 0.05 \text{ m} \).

Find \( P_L \) & \( Z_L \)!

This is one way to find an unknown load impedance \( Z_L \).

Measure SWR (using SWR meter) & locate the position of \( V_{\text{min}} \) (using slotted line), from which \( Z_L \) can be obtained as follows:

* Analytically:

  1. Find \( |P| = \frac{S-1}{S+1} = |P_L| \)

  2. Find \( \theta_L \) from \( \hat{z}_{\text{min}} \):

     \[
     \theta_L = 2\beta \hat{z}_{\text{min}} = -(2n+1)\pi
     \]

     For the 1st \( V_{\text{min}} \), \( n = 0 \)

     \[
     \Rightarrow \theta_L = 2\beta \hat{z}_{\text{min}} \quad \pi
     \]

  3. \( Z_L = Z_0 \frac{1+P_L}{1-P_L} \Rightarrow \quad P_L = |P| \ e^{j\theta_L} \)

* Using Smith Chart:

  - Draw \( |P| \) circle corresponding to \( \text{SWR} = 3 \)
  
  \( P_L \) is the location of \( V_{\text{min}} \).

  - Normalize \( \hat{z}_{\text{min}} \) to \( \lambda \)

  \[
  \frac{\hat{z}_{\text{min}}}{\lambda} = \frac{0.05}{0.4} = 0.125
  \]
we need to move 0.125 $d$ from $P_m$ towards load to locate $z_L$.

read $O$ for $P_2$ on T.Load scale

move 0.125 $d$ $\approx$ point $P_1$

connect $P_1$ with origin

note the intersection with $1\Omega$ circle, point $Q$.

$\Rightarrow z_L = 0.6 - j0.8 \Rightarrow Z_L = Z_0 z_L$

to find $P_L$

$\left| \frac{OG}{OP} \right| = 0.5$ (or use lower scale)

read $\theta_{P_L} = -90^\circ$

$\Rightarrow P_L = 0.5 \angle -90^\circ = -j0.5$
Quarter-Wave transformer.

- Used mainly to match resistive loads to a T.L.
- For maximum power transfer, it is desired that \( Z_L = Z_0 \) so that \( P = \text{zero} \).

\[
\text{For matching, need } Z_{in} = Z_0
\]

\[
\Rightarrow \frac{Z_0^2}{Z_L} = Z_0
\]

\[
\Rightarrow Z_0' = \sqrt{Z_0 Z_L}
\]

- To match \( Z_L \) to \( Z_0 \), connect a \( \frac{1}{4} \) section of a T.L. with \( Z_0' = \sqrt{Z_0 Z_L} \)

\[
\Rightarrow \text{no reflection in main line (} P = \text{zero, SWR = 1)}
\]

- Note that perfect match occurs at a single frequency, the frequency at which the section length = \( \lambda/4 \).
This is why this transformer is narrow-band. Changing the frequency ⇒ mismatch.

Example: if \( Z = 120 \, \Omega \), \( Z_0 = 75 \, \Omega \)

\[
Z'_0 = \sqrt{120 \times 75} = 95 \, \Omega
\]

What if \( Z_L \) is complex??

Remember, there are always locations on the T.L. where the input impedance is pure real (\( Z_{\text{max}} \) & \( Z_{\text{min}} \) are pure real)

\[
\Rightarrow Z'_0 = \sqrt{Z_{\text{max}} Z_0} \quad \text{or} \quad \sqrt{Z_{\text{min}} Z_0}
\]

can find \( d \) using Smith Chart.
2. Single-Stub Tuner (matching)

- Shown in the figure is a shorted shunt stub used for matching.
- Stub could be open-circuited.
- Stub could be a series stub.
- Stub has same $Z_0$ of the line. It can be different too.
- Will deal with admittance since parallel stub exists.

- Need $Y_i = 1$ (i.e., $Y_i = Y_0$)

- Find $d$ & $l$ that will give $Y_i = 1$

  $Y_i = Y_B + Y_s$ (i.e., $Y_B \parallel Y_s$)

  $\Rightarrow$ need $Y_B + Y_s = 1$
But, $y_s$ is pure imaginary since short end.

Let $y_s = -jb_s$ , $b_s \geq 0$ depending on $L$.

$\Rightarrow y_b = 1 + jb_s$

$\therefore y_b$ lies on $g=1$ circle.

Procedure:
1. Locate $z_L$ on Smith Chart
2. Draw $1\pi$-circle & locate $y_L$
3. Locate the points of intersection between the $1\pi$-circle & the $g=1$ circle
4. Find $L$
5. Find $d$ from $y_s = -jb_s$

---

Ex. 11.7

$Z_L = 40 + j30$ , $Z_0 = 100 \Omega$

Shorted shunt stub.

- $z_L = 0.4 + j0.3$
- Locate $z_L$ on Smith Chart (point P)
- Draw $1\pi$-circle & locate $y_L$ (point P')
- Locate points A & B : intersection of $1\pi$-circle with $g=1$ circle
- At A : $y_{b_2} = 1 + j1.04 \Rightarrow y_{s_2} = -j1.04$
- At B : $y_{b_1} = 1 - j1.04 \Rightarrow y_{s_1} = j1.04$
- Extend $OP'$, $OA$, & $OB$ and read from T.G. scale.
\( l_1 = (0.336 - 0.304) \lambda = 0.032 \lambda \)

\( l_2 = (0.164 + 0.5 - 0.304) \lambda = 0.36 \lambda \)

For the stub lengths:
- Locate \(-j1.04\) on Smith Chart
  \( d_2 = (0.372 - 0.25) \lambda = 0.122 \lambda \)
- Locate \(j1.04\) on Smith Chart
  \( d_1 = (0.25 + 0.128) \lambda \)
  \( d_1 = 0.378 \lambda \)
Let us consider the system shown in Fig. 7.28, in which a line is terminated by a normalized admittance \( \bar{Y}_L = (0.6 + j0.8) \) and a normalized susceptance of value \( b = 0.8 \) connected between the two conductors of the line forms the discontinuity. We wish to find the locus of the normalized admittance \( \bar{Y}_L \) to the left of the discontinuity as the susceptance slides along the line, and then determine the location, nearest to the load, of the susceptance for which the SWR to the left of it is minimized.

To construct the locus of \( \bar{Y}_L \), we first locate \( \bar{Y}_R = (0.6 + j0.8) \) on the Smith chart at point \( A \) and draw the constant SWR circle passing through \( A \), as shown in Fig. 7.29. This circle is the locus of \( \bar{Y}_R \), the normalized admittance just to the right of the discontinuity as the distance between the load and the discontinuity is varied, that is, as the susceptance slides along the line. We then choose any three points on the locus of \( \bar{Y}_R \) and locate the corresponding three points for \( \bar{Y}_L = \bar{Y}_L + j0.8 \). Here, we choose the points \( A, B, \) and \( C \). Following the constant conductance circles through these points by the amount of normalized susceptance \(-0.8\), we obtain the points \( D, E, \) and \( F \), respectively. We then draw the circle passing through these points to obtain the locus of \( \bar{Y}_L \).

Proceeding further, we note that each point on the locus of \( \bar{Y}_L \) corresponds to a value of SWR to the left of the susceptance, obtained by following the constant SWR circle through that point to the \( r \) value at the \( V_{\text{max}} \) point. In particular, it can be seen that minimum SWR is achieved to the left of the susceptance for \( \bar{Y}_R \) lying at point \( G \), which is the closest point to the center of the chart. The minimum SWR value is 1.35. The distance from the load at which the susceptance must be connected to achieve this minimum SWR can be found by locating the \( \bar{Y}_L \) corresponding to the \( \bar{Y}_R \) at \( G \) by following the constant conductance circle through \( G \) by the amount \(-0.8\) to reach point \( H \). The distance from point \( A \) to point \( H \) toward the generator is the required distance. It is equal to \((0.346 - 0.125)\lambda\), or 0.221\lambda.