1. (14 points) A capacitor consists of two coaxial metallic circular cylindrical surfaces of length $L$, inner radius $a$, and outer radius $b$. The perfect dielectric material filling between the two cylinders has a non-homogeneous permittivity:

\[ \epsilon = \epsilon_0 / \rho \]

Assume that $V(\rho = a) = V_0$ and $V(\rho = b) = 0$.

(a) (5 points) By solving the appropriate differential equation, show that the potential distribution between the two cylinders is given by

\[ V = V_0 (\rho - b) / (a - b) \]

(b) (5 points) Using the above given expression for $V$, find the capacitance of the structure using $C = Q / V$.

(c) (4 points) Check your answer in (b) by finding the capacitance using the electric energy stored in the capacitor.

2. (9 points) The space between two parallel conducting plates each having an area $S$ is filled with an inhomogeneous medium whose conductivity is given by:

\[ \sigma = \frac{2 h}{y + h} \]

where $h$ is the distance between the two plates. Assume that $V(y = 0) = 0$, and $V(y = h) = V_0$.

(a) (5 points) By solving the appropriate governing differential equation, show that the potential distribution between the two plates is given by

\[ V(y) = \frac{2 V_0}{3 h^2} \left( \frac{y^2}{2} + h y \right) \]

(b) (4 points) Using the above given expression, find the resistance between the two plates.

3. (7 points) Medium (1), comprising the region $R < a$, is a perfect dielectric sphere of permittivity $\epsilon_1$. Medium (2), comprising the region $R > a$, is free-space. The electric fields in the two regions are given by:

\[ E_1 = C_1 \left( \hat{a}_r \cos \theta - \hat{a}_\theta \sin \theta \right) \]

\[ E_2 = C_2 \left( \hat{a}_R \left( 1 + \frac{a^3}{4 R^3} \right) \cos \theta - \hat{a}_\theta \left( 1 - \frac{a^3}{4 R^3} \sin \theta \right) \right) \]

where $C_1$ and $C_2$ are constants. Find the dielectric constant $\epsilon_1$ of the dielectric sphere.
(a) \( \vec{\nabla} \cdot (\varepsilon \vec{\nabla} V) = 0 \)

\( \vec{\nabla} \cdot \left( \frac{\varepsilon_0}{\rho} \hat{\rho} \frac{dV}{d\rho} \right) = 0 \)

\( \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{\varepsilon_0}{\rho} \frac{dV}{d\rho} \right) = 0 \)

\( V = C_1 \rho + C_2 \)

\( V(\rho = a) = V_o \) \( \Rightarrow \)

\( C_1 = \frac{V_o}{a-b} \)

\( V(\rho = b) = 0 \) \( \Rightarrow \)

\( C_2 = \frac{V_o b}{b-a} \)

\( \therefore V = \frac{\rho - b}{a-b} V_o \)

(b) \( \vec{E} = -\vec{\nabla} V = -\hat{\rho} \frac{V_o}{a-b} \)

\( P_s = \varepsilon_0 E_{\rho} \bigg|_{\rho=a} = \frac{\varepsilon_0}{a} \frac{V_o}{b-a} \)

\( Q = P_s (2\pi a)(L) \Rightarrow C = \frac{Q}{V_o} = \frac{2\pi \varepsilon_0 L}{b-a} \) \( (F) \)

(c) \( W_e = \frac{1}{2} \int \varepsilon \varepsilon_0 \varepsilon V^2 d\rho \)

\( = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{b} \int_{0}^{b} \varepsilon_0 \frac{V_o^2}{(b-a)^2} \rho d\rho d\phi d\tau \)

\( = \frac{1}{2} \varepsilon_0 \frac{V_o^2}{b-a} (2\pi L) = \frac{1}{2} C V_o^2 \)

\( \Rightarrow C \) as above
(a) \[ \vec{D} \cdot (\sigma' \vec{V}) = 0 \]
\[ \vec{D} \cdot \left( \frac{2h}{y+h} \hat{y} \frac{dV}{dy} \right) = 0 \]
\[ \frac{d}{dy} \left[ \frac{2h}{y+h} \frac{dV}{dy} \right] = 0 \]
\[ V = \frac{C_1}{4h} y^2 + \frac{C_1}{2} y + C_2 \]
\[ V(y=0) = 0 \implies C_2 = 0 \]
\[ V(y=h) = V_0 \implies C_1 = \frac{4V_0}{3h} \]
\[ \therefore V = \frac{2V_0}{3h^2} \left[ \frac{y^2}{2} + yh \right] \]

(b) \[ I = \int \vec{J} \cdot d\vec{s} = \int \sigma' \vec{E} \cdot d\vec{s} \]
\[ \vec{E} = -\hat{y} \frac{dV}{dy} = -\hat{y} \frac{2V_0}{3h^2} (y+h) \]
\[ I = \int \int \left( \frac{2h}{y+h} \right) \left( \frac{2V_0}{3h^2} \right) (y+h) d\vec{s} \]
\[ = \frac{4V_0}{3h} \dot{S} \implies R = \frac{V_0}{I} = \frac{3h}{4\dot{S}} \]
At $R = a$:

$$E_{\theta_1} = E_{\theta_2} \implies C_1 = \frac{3}{4} C_2$$

& $D_{1r} = D_{2r}$

$$\implies \varepsilon_0 \varepsilon_r E_{1r} = \varepsilon_0 \varepsilon_{2r} \implies \varepsilon_r C_1 = \frac{3}{2} C_2$$

$$\implies \varepsilon_r \frac{3}{4} C_2 = \frac{3}{2} C_2 \implies \boxed{\varepsilon_r = 2}$$