1. (8 points) A particle in a central potential has an orbit angular momentum \( l = 2\hbar \), and spin \( s = 3\hbar /2 \). Find the energy levels and degeneracies associated with a spin-orbit interaction term of the form \( H_{so} = A \mathbf{L} \cdot \mathbf{S} \), where \( A \) is a constant.

**Solution:**

\[
\mathbf{L} \cdot \mathbf{S} = \frac{\hbar}{2} \left( J^2 - L^2 - S^2 \right), \quad \left| \mathbf{L} - \mathbf{S} \right| \leq J \leq \mathbf{L} + \mathbf{S}, \quad \text{and} \quad \langle \mathbf{L} \cdot \mathbf{S} \rangle = \frac{\hbar^2}{2} \left[ j(j+1) - l(l+1) \right]
\]

The degeneracy is \( 2J + 1 \), so

<table>
<thead>
<tr>
<th>( j )</th>
<th></th>
<th>Energy</th>
<th>Degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\hbar}{2} )</td>
<td>- ( \frac{9\hbar^2}{2} )</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( \frac{3\hbar}{2} )</td>
<td>- ( \frac{3\hbar^2}{2} )</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>( \frac{5\hbar}{2} )</td>
<td>( \frac{\hbar^2}{2} )</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>( \frac{7\hbar}{2} )</td>
<td>( \frac{3\hbar^2}{2} )</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

2. (8 points) Assume the particle is subjected to a uniform electric field of strength \( \varepsilon \), use first order perturbation theory to study the effect of this electric to the first four energy levels of the well.

**Solution:**

\[
\lambda H_1 = -q\varepsilon x, \quad \psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \quad E_n = \frac{n^2\hbar^2}{2mL^2}.
\]

\[
\langle \lambda H_1 \rangle = -\frac{q\varepsilon}{L} \int_0^L x \sin^2 \frac{n\pi x}{L} dx = -\frac{q\varepsilon}{L} \int_0^L \left( 1 - \cos \frac{2n\pi x}{L} \right) dx = -\frac{q\varepsilon L}{2}
\]

So, \( E_1 = \frac{\pi^2\hbar^2}{2mL^2} - \frac{q\varepsilon L}{2}, E_2 = \frac{2\pi^2\hbar^2}{mL^2} - \frac{q\varepsilon L}{2}, E_3 = \frac{4\pi^2\hbar^2}{mL^2} - \frac{q\varepsilon L}{2}, E_4 = \frac{8\pi^2\hbar^2}{mL^2} - \frac{q\varepsilon L}{2} \)

3. (9 points) A hydrogen atom is in its second ground state is subjected to an external magnetic field of 0.8 \( T \). What is the energy difference between the spin-up and spin-down states.

**Solution:**

The Hamiltonian of Hydrogen atom is \( H_0 = \frac{p^2}{2m} - \frac{e^2}{r} + \alpha L \cdot S \), since the magnetic field is strong, so we can neglect \( \alpha L \cdot S \) term, i.e., \( H_0 = \frac{p^2}{2m} - \frac{e^2}{r} \) with \( E_n = \frac{-13.6}{n^2} \) and \( \Psi = \psi_{nlm} \chi^\pm \), where \( \psi_{nlm} \) is the spatial wavefunction and \( \chi^\pm \) is the spin wavefunction. When the hydrogen atom is placed in a magnetic field, the perturbed potential is \( \lambda H_1 = \frac{eB}{2mc^2} (L_z + 2S_z) \), so \( \langle \lambda H_1 \rangle = \frac{eB}{2mc^2} (m_l + 2m_s) \). The difference in energy between spin up and spin down is

\[
E(\uparrow) - E(\downarrow) = \left( \frac{-13.6}{n^2} + \frac{eB}{2mc^2} \left( m_l + 2 \times \frac{1}{2} \right) \right) - \left( \frac{-13.6}{n^2} + \frac{eB}{2mc^2} \left( m_l + 2 \times \frac{-1}{2} \right) \right) = \frac{eB}{mc^2}
\]