8 Basic RL and RC Circuits

Goals and Objectives for this chapter:

1. Time constant for RL and RC circuits
2. Identification of Natural and forced response
3. Calculation of the total response of RL and RC circuits
4. Determination of the effect of the initial conditions on circuit response
5. Ability to express pulse waveforms using the unit step function
6. An intuitive understanding of RL and RC circuit response

8.1 Introduction

The study of transient response is an important part of system (Circuit) analysis. The analysis can show us the response of the circuit and hence how fast or slow in reaching steady state and whether oscillations occur in it and how large the transient amplitude can reach, ... etc.

8.1.1 Properties of the Exponential response

\[
\begin{array}{c|c|c|c|c|c|c}
\tau & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
e^{-\frac{t}{\tau}} & 1 & 0.37 & 0.135 & 0.05 & 0.02 & 0.007 \\
\end{array}
\]

8.1.2 The Source-Free RC Circuit

8.1.3 A More General Perspective

The first-order linear ode has the form

\[
\frac{dy}{dt} + Py = Q
\]

and its solution

\[
y(t) = Ae^{-Pt} + e^{-Pt} \int_0^t Qe^{Pu}du
\]

In transient circuits we write the ode in terms of the state variables (inductor current and capacitor voltage). Note that adding one storage element \((L \text{ or } C)\) to a circuit (where this storage element can not be combined with similar storage elements) increases the order of the ode by one.

Inductor Current

1. The voltage across an inductor \(v_L = L \frac{di}{dt}\) and at the moment of switching (transient, \(t = 0\)) the voltage required to induce a change in the inductor current is \(\infty\), hence the inductor current can not change instantly which gives the statement: \(i_L(0^-) = i_L(0^+)\)

2. In the steady state condition in DC circuits (i.e. when \(t \to \infty\)), the inductor acts as a short circuit. \((v_L = L \frac{di}{dt}\): the inductor current is constant \((\delta i_L = 0)\) and the voltage across it \(=0\), hence s.c.)

Capacitor Voltage

1. The current flowing into the capacitor \(i_C = C \frac{dv_C}{dt}\) and at the moment of switching (transient, \(t = 0\)) the current required to induce a change in the capacitor voltage is \(\infty\), hence the capacitor voltage can not change instantly which gives the statement: \(v_C(0^-) = v_C(0^+)\)

2. In the steady state condition in DC circuits (i.e. when \(t \to \infty\)), the capacitor acts as an open circuit. \((i_C = C \frac{dv_C}{dt}\): the capacitor is fully charged \((\delta v_C = 0)\) and the current flowing through it \(=0\), hence o.c.)
• General RL Circuits Consider the RL circuit shown below, for \( t \geq 0 \) we write:

\[
-V_s + Ri + L \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L}
\]

and it solution:

\[
i(t) = K_1 + K_2 e^{-t/\tau}
\]

At the moment of switching the inductor current does not change, i.e. \( i_L(0^+) = i_L(0^-) \) and in this circuit \( i_L(0^+) = 0 \). Substitute we get:

\[
0 = K_1 + K_2 \Rightarrow K_2 = -K_1
\]

When \( t \to \infty \), the inductors acts as a short circuit and \( i_L(\infty) = \frac{V_s}{R} \). Substitute we get:

\[
\frac{V_s}{R} = K_1.
\]

The current \( i_L(t) = \frac{V_s}{R}(1 - e^{-t/\tau}) \) with \( \tau = \frac{L}{R} \).

• General RC Circuits Consider the RC circuit shown below, for \( t \geq 0 \) we write:

\[
-V_s + Ri + v_C = 0 \Rightarrow -V_s + RC \frac{dv_C}{dt} + v_C = 0
\]

and its solution:

\[
v_C(t) = A + Be^{-\frac{t}{\tau}}
\]

At the moment of switching the capacitor voltage does not change, i.e. \( v_C(0^+) = v_C(0^-) \) and in this circuit \( v_C(0^+) = 0 \). Substitute we get:

\[
0 = A + B \Rightarrow B = -A
\]

When \( t \to \infty \), the capacitor acts as an open circuit and \( v_C(\infty) = V_s \) since the current flowing=0. Substitute we get:

\[
V_s = A.
\]

The voltage \( v_C(t) = V_s(1 - e^{-\frac{t}{\tau}}) \) with \( \tau = RC \).

8.1.4 Unit-Step and Delta Functions

Unit-step functions are used to describe switch opening and closing.

\[
u(t - t_0) = \begin{cases} 
0 & t < t_o \\
1 & t \geq t_o
\end{cases}
\]

The derivative of the unit-step function is the Delta function

\[
\delta(t - t_0) = \begin{cases} 
\infty & t = t_o \\
0 & t \neq t_o
\end{cases}
\]
with the property that the area under the delta function = 1, i.e.
\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \]

As a consequence the sifting property of the delta function: 
\[ \int_{-\infty}^{\infty} f(t) \delta(t - t_0) dt = f(t_0) \]
The delta function \( \delta(t - t_0) \) can be defined as the limiting case of a pulse centered at \( t = t_0 \) having width=\( w \) and height=\( h \) such that the area =\( w \times h = 1 \) when \( w \rightarrow 0 \).
A pulse can be expressed as the difference of two unit-step functions. \( u(t - t_1) - u(t - t_2) \) will result in a pulse of height=1 and width stretching from \( t_1 \) to \( t_2 \).

8.2 Driven RL Circuits
8.3 Natural and Forced Response
8.4 Driven RC Circuits
8.5 Summary and Review

- Natural response and forced response.
- Natural (transient) response depends the circuit components and the way they are configured.
- Forced response as a result of forcing function, and hence a DC forcing function will lead to a constant forced response (after 5\( \tau \) seconds).
- An RL circuit has a natural response \( i(t) = I_o e^{-t/\tau} \), where \( \tau = \frac{L}{R} \) =time constant of the RL-circuit.
- An RC circuit has a natural response \( v(t) = V_o e^{-t/\tau} \), where \( \tau = RC \) =time constant of the RC-circuit.
- Unit-step functions are used to describe switch opening and closing.
  \[ u(t - t_0) = \begin{cases} 
0 & t < t_0 \\
1 & t \geq t_0 
\end{cases} \]
  Can represent a switch that opens (or closes) at time \( t_0 \).
- The complete solution of an RL or RC circuit excited by DC source will have the form: \( f(t) = A + Be^{-t/\tau}, \) and at \( t = 0^+ \), \( f(0^+) = A + B \), at \( t = \infty \), \( f(\infty) = A \), we get \( B = f(0^+) - f(\infty) \), and the solution will be:
  \[ f(t) = f(\infty) + [f(0^+) - f(\infty)]e^{-t/\tau} \]
or: total response = final value+(initial value-final value)e^{-t/\tau}.
Example 8.6 page 275 HKD7e (with the dependent current source being 0.5 i₁):

Find $v_C(t)$ given that $v_C(0^-) = 2$ for the following circuit:

\[ \begin{align*} 
KCL \ 2: \ -i_1 + 0.5i_1 - sCv_C &= 0 \\
\text{and i}_1 &= \frac{v_2}{20+10} = \frac{v_C}{30}, \text{ combining we get} \\
-\frac{v_C}{60} - sCv_C &= 0 \\
&\Rightarrow (s + \frac{1}{60C})v_C = 0 \\
&\Rightarrow s = -\frac{1}{60C}
\end{align*} \]

and the solution $v_C(t) = A + Be^{-\frac{t}{60C}}$ and the i.c. $v_C(0^-) = v_C(0^+) = A + B = 2$ and the final condition at $t \to \infty \Rightarrow v_C(t) = 0 \Rightarrow A = 0 \Rightarrow v_C(t) = 2e^{-\frac{t}{60C}}$

Here Thevenin equivalent as seen by the capacitor is $R_{Th} = 60$

Using Maxima:

```
R:30;

i: vc(t)/30;
del1: -i+0.5*i-C*diff(vc(t),t,1)=0;
atvalue(vc(t),t=0,2); /* i.c. */
sol:desolve(del1,vc(t));
```

and the answer:

\[ v_c(t) = 2 e^{-\frac{t}{60}} \]

Practice problem 8.7/page 276 HKD7e:

Find $v_C(t)$ given that $v_C(0^-) = 11$ for the following circuit:

\[ \begin{align*} 
\text{Using Maxima:} \\
C:2e-3; \\
v1:vc(t)*(2/3); \\
de1: -vc(t)/3-1.5*v1-C*diff(vc(t),t,1)=0; \\
atvalue(vc(t),t=0,11); /* i.c. */ \\
sol:desolve(de1,vc(t));
\end{align*} \]

Answer: $v_c(t) = 11 e^{-\frac{2000}{3}}$
8.6 Irwin: key points to remember about transients in RL or RC circuits

1. The voltage or current anywhere in an RC or RL circuit is obtained by solving a first-order differential equation.

2. A solution to the equation \[ \frac{dx(t)}{dt} + \frac{x(t)}{\tau} = A \] can be written as \[ x(t) = x_{ss} + ke^{-t/\tau} \] where \( x_{ss} \) is the steady-state solution of \( x(t) \) and \( \tau \) is called the time constant of the circuit.

3. The function \( e^{-t/\tau} \) decays to a value that is less than 1% of its initial value after a period of \( 5\tau \) seconds.

4. If the circuit has a small time constant, its response to some input will quickly settle to its steady-state value; however, if the circuit time constant is large, a long time is required for the circuit to reach steady state.

5. An RC circuit has a time constant of \( R \frac{C}{T_H} \) seconds (as seen by the capacitor) and an RL circuit has a time constant of \( L \frac{R}{T_H} \) seconds (as seen by the inductor).

6. For an RL circuit, KVL: \(-v_s + Ri + L \frac{di}{dt} = 0\). Using \( s \)-operator, we can write:
   \[ Ri + sLi = v_s \Rightarrow (s + \frac{R}{L})i = \frac{v_s}{L} \]
   and the natural response can be obtained by writing the characteristic equation for the circuit which is \( s + \frac{R}{L} = 0 \),
   \[ i_n(t) = Be^{-t/\tau}, \quad \tau = \frac{L}{R} \]
   If the forcing function (excitation) is DC, then the response will be DC (constant) and the total response=natural+forced responses,
   \[ i(t) = i_n(t) + i_f(t) = Be^{-t/\tau} + A \]

7. Two methods for solving first-order circuit:
   (a) DE method
   (b) step-by-step method: use IC and SS values of state variables \( v_C(t) \) or \( i_L(t) \) together with \( \tau \) of the circuit.

8. Unit step
   \[ u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \]
   In some textbooks this is called Heaviside function \( H(t) \).

9. A pulse of width \( w \) centered at \( t - t_o \) can be written as:
   \[ p(t) = u(t + \frac{w}{2}) - u(t - \frac{w}{2}) \]
   Hence the response to a pulse is the superposition of the response to two unit steps.

10. Impulse function
    \[ \delta(t) = \frac{d}{dt} u(t) \Rightarrow \text{Area} = \int_{-\infty}^{+\infty} \delta(t)dt = 1.0 \]
    The impulse function can be viewed as the limiting case of a pulse having width \( w \) and height \( h = \frac{1}{w} \Rightarrow \), i.e. Area=\( w \times \frac{1}{w} = 1.0 \)
    \[ \delta(t) = \lim_{w \to 0, h \to \infty} \frac{p(t)}{\text{Area}=w h=1} \]

11. Sifting Property of the delta function:
    \[ \int_{-\infty}^{+\infty} f(t) \delta(t - t_o) dt = f(t_o) \]
8.7 Procedure for Evaluating Initial Conditions:


Initial values of current or voltage may be found directly from a study of the circuit schematics.

For each element in the circuit, we must determine just what will happen when the switching action takes place. For this analysis, a new schematic of an equivalent circuit for \( t = 0^+ \) may be constructed according to these rules:

1. replace all inductors with o.c. or with current sources of the value of current flowing at \( t = 0^+ \).

2. replace all capacitors with s.c. or with voltage source of the value of \( V_0 = \frac{q_0}{C} \) if there is an initial charge at \( t = 0^+ \).

3. resistors are left in the circuit without change.

8.7.1 Geometrical Interpretation of Derivative

For an RL circuit with dc supply of \( V \) volts and a switch which closes at \( t = 0 \), write kvl:

\[
\begin{align*}
-V + Ri + L \frac{di}{dt} &= 0 \\
\Rightarrow \frac{di}{dt} &= \frac{1}{L} (V - Ri) \\
\Rightarrow \frac{d^2i}{dt^2} &= \frac{-R}{L} \frac{di}{dt}
\end{align*}
\]

- \( V \) \( i \) \( L \)
- \( t = 0 \)

\[
\begin{align*}
i(t = 0^-) &= i(t = 0^+) \\
\Rightarrow \frac{di}{dt} \bigg|_{(t=0^+)} &= \frac{V}{L} \\
\Rightarrow \frac{d^2i}{dt^2} \bigg|_{(t=0^+)} &= \frac{-R}{L} \left( \frac{V}{L} \right) = \frac{-VR}{L^2}
\end{align*}
\]
Practice Problems (1-7) HKD7e, C8, solved using Maxima:

/* hkd7ec8pp.transient.mac */
/* pp8.1 */
restart; kill(all); globalsolve:true;
R:1e3; L:500e-9;
kvl: L*diff(iR(t),t,1)+iR(t)*R=0;
atvalue(iR(t),t=0,6); /* i.c. */
sol:desolve(kvl,iR(t));
tex(%);
float(subst(t=1e-9,sol));
/* answer: iR(t) = 6 e^{-2000000000 t}, iR(1.0000000000000001E-9) = 0.81201169941968 A */

/* pp8.2 */
restart; kill(all); globalsolve:true;
R1:6; R2:4; L:5; V:10;
	/* t < 0, i_L(t) = 10/4 = 2.5A */
kvl: (R1+R2)*iL(t)+L*diff(iL(t),t,1)=0;
atvalue(iL(t),t=0,2.5); /* i.c. */
sol:desolve(kvl,iL(t));
tex(%);
vL:L*diff(rhs(sol),t,1);
tex(%);
/* answer: i_L(t) = 5 e^{-2 t} and v_L(t) = -25 e^{-2 t} */

/* pp8.3 */
restart; kill(all); globalsolve:true;
iL(t):= Io*exp(-t/tau);
float(iL(2*tau)/iL(tau));
float(iL(0.5*tau)/iL(0));
float(solve(iL(t)/iL(0)=0.2,t));
float(solve(iL(0)-iL(t)=iL(0)*log(2),t));
/* answer: t=0.36787944117144, t=0.60653065971263, t=1.6094379124341 tau, t=1.181387061856003 tau */

/* pp8.4 */
restart; kill(all); globalsolve:true;
C:2e-6;
iC:*diff(vC(t),t,1);
kvl:+800*i+iC(t)=0;
atvalue(vC(t),t=0,50);
sol:desolve(kvl,vC(t));
tex(%);
float(subst(t=2e-3,sol));
/* answer: v_C(t) = 50 e^{-625 t}, v_C(0.002) = 14.3252398430095 */

/* pp8.5 */
restart; kill(all); globalsolve:true;
L:0.4; I:2; RL:2; R:8;
	/* t < 0 */
atvalue(iL(t),t=0,1.6);
	/* i_L=0.4; i_L=1.6; i_2=0; */
	/* t > 0 */
v2: 0; /* s.c. */
kvl: 2*iL(t)+L*diff(iL(t),t,1)=0;
sol:desolve(kvl,iL(t));
tex(%);
iL: float(subst(t=0.15,rhs(sol))); /* answer: \( i_L(t) = \frac{8}{5} e^{-5t} \), \( i_L(0.15) = 0.75578648438562 \) */
/* to find \( i_2 \): write kcl */
kcl: +2-i2-0-iL=0;
float(solve(kcl,i2)); /* answer: \( i_2 = 1.244213509683515 \) */

/* pp8.6 */
restart; kill(all); globalsolve:true;
/* for \( t < 0 \), we need to find \( v_C(0-) \) */
RpR(R1,R2):=R1*R2/(R1+R2);
C:4e-6;
i1:120/(1250+250);
vC0:1250*i1;
/* \( t > 0 \) */
R Th:=RpR(1250,250+600+RpR(2000,100+400));
i: -C*diff(vC(t),t,1);
kvl: -vC(t)+i*R Th;
atvalue(vC(t),t=0,vC0);
sol:desolve(kvl,vC(t));
tex(sol);
float(subst(t=1.3e-3,rhs(sol))); /* answer: \( v_C(t) = 100 e^{-400t} \), \( v_C(1.3e-3)=59.45205479701944 \) */

/* pp8.7 */
restart; kill(all); globalsolve:true;
R:3; C:2e-3;
v1:vc(t)*(2/3);
de1: -vc(t)/3-1.5*v1-C*diff(vc(t),t,1)=0;
atvalue(vc(t),t=0,11); /* i.c. */
sol:desolve(de1,vc(t));
tex(%);
/* answer: \( v_C(t) = 11 e^{-\frac{300t}{3}} \) */